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**RESEARCH ARTICLE**

# NORDHAUS-GADDUM INEQUALITIES FOR ANTI FUZZY GRAPH

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## ABSTRACT

The objective of this paper is to finds the lower and upper bounds of Nordhaus-Gaddum inequalities of fuzzy chromatic number for anti-fuzzy graph. This paper analyzes the chromatic index of complementary anti fuzzy graphs in some cases. A theorem is proved for anti-fuzzy graph to be k-critical. Examples are provided to derive the vertex coloring of these graphs.

**Keywords:** Anti fuzzy graph (AFG), Complete Anti fuzzy graph, Self-Complementary Anti fuzzy graph, Fuzzy chromatic number, Bound of N-G inequality.

## 1. INTRODUCTION

A mathematical representation helps to make out the problem in a difficult situation. The best reachable solution is to convert the difficulties into graph. Fuzzy graph theory has used to representation of many decision making problems in vague environment which have several applications. The main part of the difficulty is considered as vertices and their connection between theses vertices are considered as edges. These are represented with fuzzy value [o to 1] to determine the vagueness. Some time, vagueness exists in a relation that attains maximum value. This kind of form is known as Anti Fuzzy Graph. Graph coloring is to assign colors to certain elements of a graph subject to constraints. Vertex coloring is the most common graph coloring technique which used in various research areas of computer science such as data mining, image segmentation, clustering, image capturing and networking etc., In 1956, [7] E. A. Nordhaus and J. W. Gaddum established the inequality for the bounds of chromatic numbers to the graph and its complement when $|v(G)| = n$. Based on these results, we find the upper and lower boundary of fuzzy chromatic number to the anti-fuzzy graph and its complement. These results mainly focused by computing vertex coloring in a strong, complete and self-complementary anti fuzzy graphs.

## 2. PRELIMINARIES

Some defined definitions are list out here to focus the needed result. Undirected simple and connected graphs are considered for this work. G$(\sigma,\mu)$ is a fuzzy graph $G_A(\sigma,\mu)$ is an anti-fuzzy graph with primary set S. S is a fuzzy subset of non-empty set is a mapping $\sigma: S \to [0,1]$, and a fuzzy relation $\mu$ on fuzzy subset $\sigma$, is a fuzzy subset of $S \times S$.

A Fuzzy Graph G$(\sigma,\mu)$ with $\sigma: S \to [0,1]$ and $\mu: S \times S \to [0,1]$ *such that* $\mu$ (x,y) ≤ min$(\sigma(x),\sigma(y))$ $\forall$ x, y in S.

Anti-Fuzzy Graph $G_A(\sigma,\mu)$ consists of $\sigma: S \to [0,1]$ & $\mu: S \times S \to [0,1]$ such that $\mu$(x,y) ≥ max$(\sigma(x),\sigma(y))$ $\forall$ x, y in S. The degree of a vertex $\sigma(x)$ of an anti-fuzzy graph $G_A(\sigma,\mu)$ is defined by, $d_{G_A}(\sigma(x)) = \Sigma_{x \neq y}\, \mu(x,y)$. $G_A(\sigma,\mu)$ is called strong if $\mu$(x,y) = max$(\sigma(x),\sigma(y))$ for all



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(x,y) in $\mu$. $G_A(\sigma,\mu)$ is called complete if $\mu$(x,y) = $\max(\sigma(x),\sigma(y))$ for all x, y in $\sigma$.

The complement of Anti fuzzy graph $\bar{G}_A(\bar{\sigma},\bar{\mu})$ is derived from an fundamental Anti fuzzy graph $G_A(\sigma,\mu)$ where $\sigma = \bar{\sigma}$ and $\bar{\mu}$ (x,y) = $\mu$ (x,y) – $\max(\sigma(x),\sigma(y))$, if $\mu$(x,y) > 0. If $G_A = \overline{G_A}$ then $G_A$ is said to be a self-complementary anti fuzzy graph.

In a crisp graph, a graph is said to be k-critical if, $\chi(H) < \chi(G)$ for every proper sub graph H of G. A k-Chromatic graph that is critical is called k-critical. Every k – chromatic graph has a k-critical sub graph. Vertex coloring is mapping $\beta: V \rightarrow N$ such that $\beta(i) \neq \beta(j)$ where i and j are the adjacent vertices in G. A graph is k-colourable if it admits k-colorings. The chromatic number $\chi(G)$, of a graph G is the minimum k for which G is k-colorable.

The fuzzy vertex coloring of a fuzzy graph was defined by [10]Eslahchi and Onagh. If a fuzzy graph is k-fuzzy coloring then it satisfied the follows.

i). Let $\Delta = \{\gamma_1,\dots\gamma_k\}$ be the k- colouring such that $\cup\Delta = \sigma$

ii). $\gamma_i \wedge \gamma_j = 0$ and

iii). For every adjacent vertices u,v of G, min $\{\gamma_i(u),\gamma_i(v)\}$=0 (1≤ i ≤ k).

[11] M.A. Rifayathali, A.Prasanna and S.Ismail Mohideen defined the coloring and chromatic number of anti-fuzzy graphs using β-cuts. A set of $\beta$-cut, generated by an anti-fuzzy set A, where $\beta \in [0,1]$ is a fixed numbers, is defined as $A_\beta = \{x \in X \,/\, \mu_A(x) \leq \beta, \forall\beta \in [0,1]\}$.

The family of $\beta$-cut sets $A_\beta$ is monotone, that is for $\beta, \delta \in [0,1]$ and $\beta \geq \delta$ and $A_\beta \supseteq A_\delta$.

## 3. CHROMATIC INDEX OF ANTI FUZZY GRAPH

[11] Definition 3.1: Let $\{G_\beta = (V_\beta, E_\beta)/\beta \in [0,1]\}$ be the family of $\beta$-cuts of $G$, where the $\beta$-cut of an anti- fuzzy graph is the crisp graph $G_\beta = (V_\beta, E_\beta)$ with $V_\beta = \{v_i/v_i \in V, \sigma(v_i) \leq \beta\}$, $E_\beta = \{(v_iv_j)/(v_iv_j) \in E, \mu(v_iv_j) \leq \beta\}$.

To determine chromatic number of AFG, family of $\beta$- cuts of $G$ is determined and then the chromatic number $\chi_\beta$ of each $G_\beta$ is determined by using crisp k- vertex coloring $C_\beta^k$. The Chromatic number of $G$ is defined through a monotone family of sets.

[11] Definition 3.2 : For an anti-fuzzy graph $G = (V, E)$, its chromatic number is the anti-fuzzy number $\chi(G) = \{(x, k(x))/x \in X\}$, where $X = \{1, \dots, |V|\}$, $k(x) = \inf\{\beta \in [0,1]/x \in A_\beta\}$, $x \in X$ and $A_\beta = \{\chi_0, \dots, \chi_\beta\}$, $\beta \in [0,1]$.

## 4. CHROMATIC NUMBER OF SELF COMPLEMENTARY ANTI FUZZY GRAPH

This section observes the features of self-complementary anti fuzzy graph and derives the fuzzy chromatic number for complement of anti-fuzzy graph.

**Proposition 4.1**

For a self-complement AFG, the underlying graph and its complement must be strong. But every strong AFG need not be self-complementary.

**Proposition 4.2**

If self-complementary exists in AFG then it must be a v-nodal anti fuzzy graph with effective edges.

Proof: Every n-vertex self-complementary graph has exactly n(n-1)/4 edges and it must have the diameter either 2 or 3. [5]In complementary AFG, the vertices are isomorphic to each other by the property of complement. That is, $\sigma = \bar{\sigma}$.

if $\mu$(x,y) > 0,

$\bar{\mu}$(x,y) = $\mu$(x,y) – $\max(\sigma(x),\sigma(y))$ .

To claim the isomorphism, $\mu$(x,y) must be equal to 0. From this,

$\bar{\mu}$(x,y) = $\max(\sigma(x),\sigma(y))$.

Every (3x3 rook's graph) Paley graph is the best example of self-complementary graphs.

Example 4.5 with $\sigma(G_A)$ = 0.8 for all $\sigma$ provides the self-complement.

**Proposition 4.3**

A Self complementary graph doesn't exist in a complete anti fuzzy graph. Since the complement of complete anti fuzzy graph is a null graph.

There is no possible to finds an isomorphism between them.

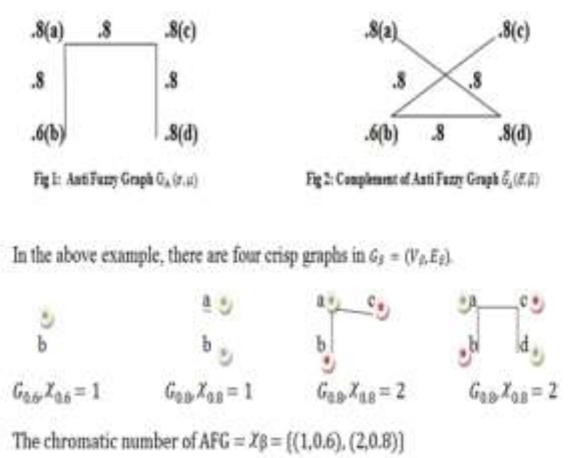
**Proposition 4.4**

If AFG preserves the self-complement, then their chromatic number are same. Converse part of this statement is not necessarily true.

Proof:

If AFG preserves self-complement then, $G_A(\sigma,\mu) = \bar{G}_A(\bar{\sigma},\bar{\mu})$. That is, there exists an isomorphism between $\sigma$ and $\mu$. obviously, $\chi(G_A) = \chi(\bar{G}_A)$.

**Example-4.5** Computation of fuzzy chromatic index using the $\beta$-cuts




That is their chromatic index,

$\chi(G_A) = \chi(\bar{G}_A) = \{(1, 0.6), (2, 0.8)\}$.

## 5. CHROMATIC NUMBER OF COMPLETE ANTI FUZZY GRAPH AND ITS COMPLEMENT

This section observes the chromatic number of a complete AFG and compared this result with its complementary graph. From this observation, a circumstance is provided for the computation of chromatic number.

**Proposition 5.1**

In AFG, the following observation holds.

1. For every vertex graph, chromatic number by vertex coloring is 1.
2. For every edge graph, chromatic number (vertex coloring) is 2. Since the vertices are adjacent.
3. For any AFG, $1 \le \chi(G_A) \le$ n.

**Proposition 5.2**

For $|\sigma(G_A)| = n,$ anti-fuzzy graph, the following relation holds.

1. Fuzzy chromatic number

   $\chi(G_A)$ = n, if AFG is complete.

   $\chi(G_A)$ < n, otherwise

2. If $G_A$ is complete then,

   $\chi(\bar{G}_A) = 1$ and $\chi(G_A) > \chi(\bar{G}_A)$.

3. $G_A$ is k-critical graph when *it* is complete.

*Proof*:

1. For a complete AFG, all the edges are effective for all $\sigma$ in $G_A$. Hence every vertex needs to assign distinct colors. That is, $\chi(G_A)$ = n. If AFG is not complete then at least one of edge is not effective. ie., $\chi(G_A)$ < n.
2. Complement of complete FG does not have edges. That is, all the vertices are distinct. Clearly, $\chi(\bar{G}_A) = 1$. From 1 &2, we concluded that, the chromatic number of complete AFG is finer than its complement.
3. In complete AFG, we cannot obtain the self-complementary. There is no possible to exist isomorphism between $\chi(G_A)$ and $\chi(\bar{G}_A)$. This implies that $\bar{G}_A$ is a subgraph of Anti Fuzzy Graph.

This says that, $\chi(\bar{G}_A) < \chi(G_A)$. Hence, $G_A$ is k-critical graph.

**Proposition 5.3**

A complete AFG with 'n' vertices and its complement satisfies,

1. $\chi(G_A)$ = n+1 - $\chi(\bar{G}_A)$

2. $\chi(G_A) = \frac{n}{\chi(\bar{G}_A)}$

*Proof*:

By proposition 5.2, if AFG is complete then, $\chi(G_A)$ = n and $\chi(\bar{G}_A)$ = 1. This says that, $\chi(G_A) + \chi(\bar{G}_A)$ = n+1 and $\chi(G_A) \cdot \chi(\bar{G}_A)$ = n.

**Proposition 5.4**

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The chromatic number of any AFG $G_A$ with 'n' vertices and its complement $\bar{G}_A$ satisfies,

1. $n+1 \le \chi(G_A) + \chi(\bar{G}_A) \le 2n$
2. $n \le \chi(G_A) \cdot \chi(\bar{G}_A) \le n^2$

*Proof*:

Consider any AFG need not be strong or complete. Take every vertices is adjacent to each other with anti-fuzzy values. Then $\chi(G_A)$ = n. For any edge membership value $\chi(\bar{G}_A)$ is not more than 'n'. That is, $\chi(\bar{G}_A) \le n$. This implies that,

$$\chi(G_A) + \chi(\bar{G}_A) = n + (\le n) \le 2n$$

$$\chi(G_A) \cdot \chi(\bar{G}_A) = n \cdot (\le n) \le n^2$$

**Example: 5.5**

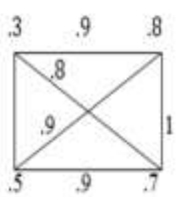


Fig 3: Anti Fuzzy Graph $G_A(\sigma,\mu)$

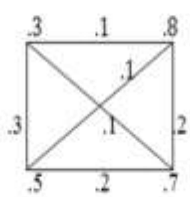


Fig 4: Complement of Anti Fuzzy Graph $\bar{G}_A(\sigma,\mu)$

From this example,

$\chi(G_A) = \chi(\bar{G}_A) \rightarrow \chi(G_A) + \chi(\bar{G}_A) \le 2n$

$\rightarrow \chi(G_A) \cdot \chi(\bar{G}_A) \le n^2$

**5. CONCLUSION**

The chromatic number of an anti-fuzzy graph is derived based on vertex coloring. This paper concludes that the chromatic number of self-complementary graph is same. The sum and product of chromatic number of complete anti-fuzzy graph is determined. For $|G_A(\sigma, \mu)| = n$, lower and upper bounds for the Nordhaus - Gaddum inequality is, $n+1 \le \chi(G_A) + \chi(\bar{G}_A) \le 2n$ and $n \le \chi(G_A) \cdot \chi(\bar{G}_A) \le n^2$.

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