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ABSTRACT

A b-coloring of a graph is a proper coloring where each color admits at least one node (called dominating node) adjacent to every other used color. The maximum number of colors needed to b-color a graph G is called the b-chromatic number and is denoted by \( \phi(G) \). In this paper, we find the b-chromatic number and some of the structural properties of corona product of crown graph and complete bipartite graph with path graph.

Keywords: Corona product, crown graph, complete bipartite graph, path graph.

1. INTRODUCTION

A b-coloring by k-colors is a proper coloring of the vertices of graph G such that in each color classes there exists a vertex that has neighbors in all the other k-1 color classes. The b-chromatic number \( \phi(G) \) is the largest number \( k \) for which G admits a b-coloring with k-colors (Irving and Manlove, 1999). The corona \( G_1 \odot G_2 \) of two graphs \( G_1 \) and \( G_2 \) is defined as a graph obtained by taking one copy of \( G_1 \) (which has \( p_1 \) vertices) and \( p_1 \) copies of \( G_2 \) and attach one copy of \( G_2 \) at every vertex of \( G_1 \) (Harary, 1972).

In this paper we find for which the largest number \( k \) for which corona product of crown graph and complete bipartite graph with path graph admits a b-coloring with k-colors. And also we find some of its structural properties (Venkatachalam and Vernold Vivin, 2010; Vernold Vivin and Venkatachalam, 2012; Vijayalakshmi and Thilagavathi, 2012)

2. DEFINITION

2.1. Crown Graph

A crown graph on \( 2n \) vertices is an undirected graph with two sets of vertices \( u_i \) and \( v_i \) and with an edge from \( u_i \) to \( v_j \) whenever \( i \neq j \). The crown graph can be viewed as a complete bipartite graph from which the edges of a perfect matching have been removed (Wikipedia).

2.2. Complete Bipartite Graph

A complete bipartite graph is a graph whose vertices can be partitioned into two subsets \( V_1 \) and \( V_2 \) such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. That is, it is a bipartite graph \( (V_1, V_2, E) \) such that for every two vertices \( v_1 \in V_1 \) and \( v_2 \in V_2 \), \( v_1v_2 \) is an edge in \( E \). A complete bipartite graph with partitions of size \( |V_1|=m \) and \( |V_2|=n \) is denoted \( K_{mn} \) (Balakrishnan, 2004; Balakrishnan and Ranganathan, 2012).

2.3. Fan Graph

A Fan graph \( F_{m,n} \) is defined as the graph join \( K_m \) and \( P_n \) is the path on \( n \) nodes (Wikipedia).

2.4. Path Graph

The path graph \( P_n \) is a tree with two nodes of vertex degree 1, and the other \( n-2 \) nodes of vertex degree (Harary, 1972).

2.5. Corona Product

Corona product or simply corona of any graph \( G_1 \) and graph \( G_2 \), defined as the graph which is the disjoint union of one copy of \( G_1 \) and \( |V_1| \) copies of \( G_2 \) \( (|V_1| \) is the number of vertices of \( G_1 \)) in which each vertex of the copy of \( G_1 \) is connected to all vertices of a separate copy of \( G_2 \) (Harary, 1972).

2.6. b-coloring

A b-coloring of a graph is a proper coloring such that every color class contains a vertex that is adjacent to all other color classes. The b-chromatic number of a graph \( G \), denoted by \( \phi(G) \), is the maximum number \( t \) such that \( G \) admits a b-coloring with \( t \) colors (Irving and Manlove, 1999).
3. CORONA PRODUCT OF CROWN GRAPH WITH PATH GRAPH

3.1. b-chromatic number of corona product of Crown Graph with Path Graph

3.1.1. Theorem

For any n ≥ 3, \( \varphi[S_n^0 \circ P_n] = 2n \).

Proof: Let \( S_n^0 \) be any Crown graph with vertices, \( V = \{v_1, v_2, \ldots, v_n\} \) and \( V = \{v', v_1, v_2, \ldots, v_n\} \). i.e. \( V(S_n^0) = V U V' \).

Let the edges of \( L_n \) be \( E(S_n^0) = \{e_j: 1 < j < n^2 - n\} \) where \( e_j \) is the edge connecting \( v_i \) and \( v_j \) for every \( i \neq j \).

Let \( P_n \) be any path graph of length \( n-1 \) with \( n \)-vertices. \( V(P_{n-1}) = \{v_0, v_1, \ldots, v_n\} \) and \( E(P_n) = \{e_{01}, e_{12}, \ldots, e_{(n-1)n}\} \).

By the definition of corona graph each vertex in \( S_n^0 \) is adjacent to every vertex copy of \( P_n \), i.e. vertices of \( V(S_n^0) \cup V(P_n) \). Let \( E[S_n^0 \circ P_n] = E(S_n^0) U E(P_n) \).

Consider the color class \( C = \{c_1, c_2, \ldots, c_n, c_{n+1}, c_{n+2}, \ldots, c_{2n}\} \) to color the vertices of \( S_n^0 \). Assign the colors \( c_1, c_2, \ldots, c_n \) to the vertices of \( S_n^0 \) and \( c_{n+1}, c_{n+2}, \ldots, c_{2n} \) to \( v_j \), \( j = 1, 2, \ldots, n \).

From the figure we see that, each \( v_i \) and \( v_j \) are adjacent to every \( v_j \) for every \( i \) not equal to \( j \) and vice versa. Hence both \( v_i \) and \( v_j \) earns its adjacent color for every \( i \neq j \). To make the above coloring to be \( b \)-chromatic proper coloring of \( V(P_n) \) by corresponding non-adjacent vertices of its \( v_j \) and \( v_j \) respectively. Thus each color has the neighbor in the every other color class. Thus \( \varphi[S_n^0 \circ P_n] = 2n \).

Let us assume that \( \varphi[S_n^0 \circ P_n] = 2n \). The graph \( S_n^0 \circ P_n \) must requires \( 2n+2 \) vertices of degree \( 2n+1 \), all with distinct color and each must have adjacent with all of the other color class, but at least one color class which does not have a color dominating vertex in \( S_n^0 \circ P_n \), which invalidates the definition of \( b \)-coloring. Hence, \( \varphi[S_n^0 \circ P_n] \) is not equal to \( 2n+1 \), it must be less than \( 2n+1 \) i.e. \( \varphi[S_n^0 \circ P_n] = 2n \). Thus, for any \( n \geq 3 \), the \( b \)-chromatic number of corona graph of crown graph with path graph is \( 2n \).

3.2. Illustration: \( b \)-coloring of corona product of Crown Graph with Path Graph

3.2.1. Theorem

For any \( n \geq 3 \), \( \varphi[S_n^0 \circ P_n] = 5n^2 - 3n \)

Proof: \( \varphi[S_n^0 \circ P_n] = \) Number of edges in \( S_n^0 + 2n x \)

Theorem:

For any \( n \geq 3 \), \( \varphi[S_n^0 \circ P_n] = 5n^2 - 3n \).

Proof: \( \varphi[S_n^0 \circ P_n] = \) Number of edges in \( S_n^0 \)

4. CORONA PRODUCT OF COMPLETE BIPARTITE GRAPH WITH PATH GRAPH

4.1. b-chromatic number on Corona Product of Complete Bipartite Graph with Path Graph
4.1.1. Theorem

For any \( n \geq 3 \), \( \varphi [K_{m,n} \circ P_n] \) \( \neq 2n+1 \)

\[
K_{m,n} = \begin{cases} \text{2n+1} & \text{m > n} \\ \text{2n-1} & \text{m < n} \end{cases}
\]

Proof: Let \( K_{m,n} \) be any complete bipartite graph with vertices \( V = \{v_1, v_2, \ldots, v_m\} \) and \( P = \{p_1, p_2, \ldots, p_n\} \) i.e.,

\[
V(K_{m,n}) = V \cup P. \text{ Let the edges } E(K_{m,n}) = \{e_i : 1 \leq i \leq \binom{m+n}{2}\} \text{ where } e_i \text{ is the edge connecting } v_i \text{ and } v_j.
\]

Let \( P_n \) be any path graph of length \( n-1 \) with \( n \)-vertices. \( V(P_n) = \{\mathcal{U} : 1 \leq i \leq 2n, 1 \leq j \leq n, 1 \leq f \leq n \} \) and \( E(P_n) = \{e_i : 2n-1 \leq i \leq n-1\} \).

By the definition of corona graph each vertex in \( K_{m,n} \) is adjacent to every vertex copy of \( P_n \), i.e. vertices of \( V(K_{m,n} \circ P_n) = V(K_{m,n}) \cup V(P_n) \).

Let the edges of \( K_{m,n} \circ P_n \) be \( E[K_{m,n} \circ P_n] = E(K_{m,n}) \cup E(P_n) \{e_i : 2n+1 \leq i \leq \binom{m+n}{2}\} \) for \( m > n \). \( E[K_{m,n} \circ P_n] = E(K_{m,n}) \cup E(P_n) \{e_i : 2n \leq i \leq \binom{m+n}{2}\} \) for \( m < n \).

Consider the color class \( C_1 = \{c_1, c_2, c_3, \ldots, c_n, c_{n+1}, c_{n+2}, \ldots, c_{2n}\} \) to color the vertices of \( K_{m,n} \circ P_n \). The proof follows from the following cases.

Case (i) \( m=n \)

Consider the color class \( C_2 = \{c_1, c_2, c_3, \ldots, c_n, c_{n+1} \} \) to color the vertices of \( K_{m,n} \circ P_n \). Assign the colors \( c_1, c_2, c_3, \ldots, c_n \) to \( v_1, v_2, \ldots, v_n \), i.e. \( v_i \cdot s \) and \( c_{n+1}, c_{n+2}, \ldots, c_{2n-1} \) to \( v_j \cdot s \), \( j = 1, 2, 3, \ldots, m \) respectively.

The remaining proof of the theorem follows immediately from case (i). Hence \( \varphi [K_{m,n} \circ P_n] = 2n+1, m > n \).

Case (ii) \( m=n \)

Consider the color class \( C_3 = \{c_1, c_2, c_3, \ldots, c_n, c_{n+1} \} \) to color the vertices of \( K_{m,n} \circ P_n \). Assign the colors \( c_1, c_2, c_3, \ldots, c_n \) to \( v_1, v_2, \ldots, v_n \), i.e. \( v_i \cdot s \) and \( c_{n+1}, c_{n+2}, \ldots, c_{2n-1} \) to \( v_j \cdot s \), \( j = 1, 2, 3, \ldots, m \) respectively.

The remaining proof of the theorem follows immediately from case (i). Hence \( \varphi [K_{m,n} \circ P_n] = 2n-1, m < n \). Hence the proof.

4.1.1. Illustration Corona Product of Complete Bipartite Graph with Path Graph

4.1.2. Theorem

For any \( m, n \geq 3 \), \( q[\{K_{m,n} \circ P_n\}] = 2n^2+3mn-mn \).

Proof: \( q[S_{m,n}] = \text{Number of edges in } K_{m,n} \circ P_n \)

Number of edges in Fan graph \( F_n \)

\[
q[S_{m,n}] = mn+ (m+n) (2n-1) - mn+2mn-m+2n^2-n = 2n^2+3mn-mn
\]
4.1.3. **Theorem**

For any \( m, n \geq 3 \), the vertex polynomial of be( \( K_{m,n} \circ P_n \)) \( 4nx^2 + (2n^2 - 4n)x^3 + 2nx^{2n} \), \( m=n \).

**Proof:** \( V(K_{m,n} \circ P_n ; x) = \Delta(G) V x^k \)

= No of vertices having degree \( 2 \times x^2 \) +

No of vertices having degree \( 3 \times x^3 \) +

No of vertices having degree \( 2n \times x^{2n} \) +

= \( 4nx^2 + 2n^2 - 4n x^3 + 2n x^{n+2} \).

4.1.4. **Theorem**

4.4.5. **Some Structural Properties of \((K_{m,n} \circ P_n)\), \( m \geq 3 \).**

<table>
<thead>
<tr>
<th>Properties</th>
<th>Number of Vertices</th>
<th>Number of Edges</th>
<th>Maximum Degree</th>
<th>Minimum Degree</th>
<th>Vertex Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>Path Graph</td>
<td>( n )</td>
<td>( n-1 )</td>
<td>2</td>
<td>1</td>
<td>( 2x+(n-2)x^2 )</td>
</tr>
<tr>
<td>Complete Bipartite Graph</td>
<td>( m+n )</td>
<td>( n^2 )</td>
<td>( \text{max}(m,n) )</td>
<td>( \text{min}(m,n) )</td>
<td>( (2n)x^n )</td>
</tr>
<tr>
<td>( (K_{m,n} \circ P_n) ) ( m = n )</td>
<td>( 2n^2 + 2n )</td>
<td>( 5n^2 - 2n )</td>
<td>2n</td>
<td>2</td>
<td>( 4nx^2 + (2n^2 - 4n)x^3 + 2nx^{2n} )</td>
</tr>
<tr>
<td>( (K_{m,n} \circ P_n) ) ( m &lt; n )</td>
<td>( 2n^2 )</td>
<td>( 5n^2 )</td>
<td>2n</td>
<td>2</td>
<td>( (4n - 2)x^2 + (2n^2 - 3n - 2)x^3 ) + ( 2n + 1 ) ( x^{2n+1} )</td>
</tr>
<tr>
<td>( (K_{m,n} \circ P_n) ) ( m &gt; n )</td>
<td>( 2n^2 )</td>
<td>( 5n^2 )</td>
<td>2n+1</td>
<td>2</td>
<td>( (4n + 2)x^2 ) + ( (2n^2 - 3n - 2)x^3 ) + ( 2n + 1 ) ( x^{2n+1} )</td>
</tr>
</tbody>
</table>

5. **CONCLUSION**

In this paper we operated the graph operation corona product on crown graph and complete bipartite graph with path graph, we get corona product of crown graph with path graph and corona product of complete bipartite graph with path graph and also we find its b-chromatic number and some of its structural properties.

**REFERENCES**


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