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ON $\alpha\delta^{\theta\#}\mathcal{C}$ AND $\alpha\delta^{\theta\#}\mathcal{A}$ IN TOPOLOGICAL SPACES

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ABSTRACT

In (Devi *et al*., 2012), the authors introduced the notion of $\alpha\delta$-closed sets and investigated its fundamental properties. In this paper, we investigate some more properties of this type of closed spaces.

Keywords: $\alpha\delta^{\theta\#}$-convergence and $\alpha\delta^{\theta\#}$-adherence.

1. INTRODUCTION

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized open sets and the idea of grills on a topological space was first introduced by Choquet. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds.

Throughout the present paper, spaces X and Y always mean topological spaces. Let X be a topological space and A a subset of X. For a subset A of a topological space (X, τ), $cl(A)$ and $int(A)$denote the closure of A and the interior of A, respectively. A subset A is said to be regular open (resp. regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$, The δ-interior of a subset A of X is the union of all regular open sets of X contained in A and is denoted by $Int_\delta(A)$.The subset A is called δ-open if $A = Int_\delta(A)$, i.e., a set is δ-open if it is the union of regular open sets. The complement of a δ-open set is called δ-closed. Alternatively, a set$A \subset X, \tau$ is called δ-closed if $A = cl_\delta(A)$, where $cl_\delta\ A\ = \mathrm{x/x} \in \mathrm{U} \in \tau \Rightarrow int(cl(A\ \ \cap A \neq \varphi$. The family of all δ-open (resp. δ-closed) sets in X is denoted by $\delta O(X)$ (resp. $\delta C(X)$).

A subset A of X is called semiopen (Kokilavani and Basker, 2012d) (resp. α-open (Roy and Mukherjee, 2009), δ-semiopen) if $A \subset cl(int(A))$ (resp.$A \subset int(cl(int\ A\ ))$, $A \subset cl(Int_\delta\ (A)))$ and the complement of a semiopen(resp. α-open, δ-semiopen) are called semiclosed(resp. α-closed, δ-semiclosed). The intersection of all semiclosed (resp. α-closed, δ-semiclosed) sets containing A is called the semi-closure(resp. α-closure, δ-semiclosure) of A and is denoted by $scl(A)$(resp.$\alpha cl(A)$, δ-$scl(A)$). Dually, semi-interior(resp. α-interior, δ-semi-nterior) of A is defined to be the union of all semiopen (resp. α-open,δ-semiopen) sets contained in A and is denoted by $sint(A)$(resp.$\alpha int\ (A)$, δ-$sint(A)$). Note that δ-$scl\ A\ = A\ \ int(cl_\delta(A))$ and δ-$sint(A) = A\ \ cl(Int_\delta\ (A))$.In (Devi *et al*., 2012), the authors introduced the notion of $\alpha\delta$-closedspaces and investigated its fundamental properties. In this paper, we investigate some more properties of this type of closed spaces.

Before entering to our work, we recall the following definitions, which are useful in the sequel.

Definition 1.1. (Devi *et al*., 2012) A subset A of a space X is said to be

(a) An α-generalized closed (αg-closed) set if $\alpha cl(A)\ \subseteq U$whenever $A \subseteq U$ and U is α-open in (X, τ).
(b) A $\alpha\delta$-closed set if $\underset{\delta}{cl}(A) \subseteq U$whenever $A \subseteq U$ and U is αg-open in (X, τ).

2. $\alpha\delta^{\theta\#}$-CONVERGENCE AND$\alpha\delta^{\theta\#}$-ADHERENCE

Definition 2.1. (Choquet, 1947) A grill $\mathcal{G}$on a topological space $(X, \tau)$is defined to be a collection of nonempty subsets of Xsuch that $(i)\ A \in G$ and $A \subset B \subset X \Rightarrow B \in G$ and $ii\ \ A, B \subset X$and $A \cup B \in G \Rightarrow A \in G\ or\ B \in G$.

Definition 2.2. (Choquet, 1947) If $\mathcal{G}$is a grill (or a filter) on a space (X, τ), then the section of $\mathcal{G}$, denoted by $sec\,\mathcal{G}$, is given by $sec\,\mathcal{G} = \{\ A \subseteq X: A \cap G \neq \varphi\ , for\ all\ G \in \mathcal{G}\}$.

Definition 2.3. A grill $\mathcal{G}$ on a topological space (X, τ) is said to be an:

(a) $\alpha\delta^{\theta\#}$-adhere (briefly. $\alpha\delta^{\theta\#}\mathcal{A}$) at $x \in X$ if for each $U \in \alpha\delta O(x)$ and each $G \in \mathcal{G}$,$\alpha\delta_{Cl} U \cap G \neq \varphi$,

(b) $\alpha\delta^{\theta\#}$-converge (briefly. $\alpha\delta^{\theta\#}\mathcal{C}$) to a point $x \in X$ if for each $U \in \alpha\delta O(x)$, there is some $G \in \mathcal{G}$ such that $G \subseteq \alpha\delta_{Cl} U$ (in this case we shall also say that $\mathcal{G}$ is $\alpha\delta^{\theta\#}$-convergent to x).

Remark 2.4. A grill $\mathcal{G}$ is $\alpha\delta^{\theta\#}\mathcal{C}$ to a point $x \in X$ if and only if $\mathcal{G}$contains the collection $\{\alpha\delta_{Cl} U : U \in \alpha\delta O(x)\}$.

Definition 2.5. A filter $\mathcal{F}$on a space (X, τ) is said to $\alpha\delta^{\theta\#}\mathcal{A} x \in X$ ($\alpha\delta^{\theta\#}\mathcal{C}$ to $x \in X$) if for each $F \in \mathcal{F}$and each $U \in \alpha\delta O(x)$, $F \cap \alpha\delta_{Cl}\ U \neq \varphi$ (resp. to each $U \in \alpha\delta O(x)$, there corresponds $F \in \mathcal{F}$such that $F \subseteq \alpha\delta_{Cl} U$.

We note at this stage that unlike the case of filters, the notion of $\alpha\delta^{\theta\#}\mathcal{A}$ of a grill is strictly stronger than that of $\alpha\delta^{\theta\#}\mathcal{C}$. In fact, we have

Theorem 2.6. If a grill $\mathcal{G}$on a space(X, τ), $\alpha\delta^{\theta\#}\mathcal{A}$ at some point $x \in X$, then $\mathcal{G}$ is $\alpha\delta^{\theta\#}\mathcal{C}$ to x.

Proof. Let a grill $\mathcal{G}$ on (X, τ), $\alpha\delta^{\theta\#}\mathcal{A}$ at $x \in X$. Then for each $U \in \alpha\delta O\ x$ and each $G \in \mathcal{G}$, $\alpha\delta_{Cl} U \cap G \neq \varphi$ so that $\alpha\delta_{Cl} U \in sec\ \mathcal{G}$,for each $U \in \alpha\delta O(x)$,and hence $X - \alpha\delta_{Cl} U \notin \mathcal{G}$. Then $\alpha\delta_{Cl}\ U \in \mathcal{G}$(as $\mathcal{G}$ is a grill and $X \in \mathcal{G}$), for each $U \in \alpha\delta O(x)$. Hence $\mathcal{G}$ must $\alpha\delta^{\theta\#}\mathcal{C}$ to x.

The following example shows that a $\alpha\delta^{\theta\#}\mathcal{C}$ grill need not $\alpha\delta^{\theta\#}\mathcal{A}$ at any point of the space even if the space is finite.

Example 2.7. Let $X = \{a, b, c\}$ and $\tau = \varphi, X, a\ , b\ , a, b\ , a, c$ It is easy to verify that (X, τ) is a topological space such that $\alpha\delta O\ X, x\ = \tau$. Let $\mathcal{G} = b\ , a, b\ , b, c\ , X$. Then $\mathcal{G}$ is $\alpha\delta^{\theta\#}\mathcal{C}$ but not $\alpha\delta^{\theta\#}\mathcal{A}$.

Remark 2.8. Let X be a topological space. Then for any $x \in X$, we adopt the following notation:

$\mathcal{G}\ \alpha\delta^{\theta\#}, x = \{A \subseteq X: x \in \alpha\delta^{\theta\#}cl(A)\}$,

$sec\ \mathcal{G}\ \alpha\delta^{\theta\#}, x = \{\ A \subseteq X: A \cap G \neq \varphi\ , for\ all\ G \in \mathcal{G}\ \alpha\delta^{\theta\#}, x\ \}$.

In the next two theorems, we characterize the $\alpha\delta^{\theta\#}\mathcal{A}$ and $\alpha\delta^{\theta\#}\mathcal{C}$ of grills in terms of the above notations.

Theorem 2.9. A grill $\mathcal{G}$ on a space (X, τ), $\alpha\delta^{\theta\#}\mathcal{A}$ to a point $x \in X$ if and only if $\mathcal{G} \subseteq \mathcal{G}\ \alpha\delta^{\theta\#}, x$.

Proof. A grill $\mathcal{G}$ on a space (X, τ). $\alpha\delta^{\theta\#}\mathcal{A}$ at $x \in X$

$\Rightarrow \alpha\delta_{Cl}\ U \cap G \neq \varphi\ for\ all\ U \in \alpha\delta O\ x\ and\ all\ G \in \mathcal{G}$

$\Rightarrow x \in \alpha\delta^{\theta\#}cl\ G\ , for\ all\ G \in \mathcal{G}$

$\Rightarrow G \in \mathcal{G}\ \alpha\delta^{\theta\#}, x\ , for\ all\ G \in \mathcal{G}$

$\Rightarrow \mathcal{G} \subseteq \mathcal{G}\ \alpha\delta^{\theta\#}, x$.

Conversely, let $\mathcal{G} \subseteq \mathcal{G}\ \alpha\delta^{\theta\#}, x$. Then for all $G \in \mathcal{G}$, $x \in \alpha\delta^{\theta\#}cl\ G$, so that for all $U \in \alpha\delta O\ x$ and for all $G \in \mathcal{G}$, $\alpha\delta_{Cl}\ U\ \cap G \neq \varphi$. Hence $\mathcal{G}$ is $\alpha\delta^{\theta\#}\mathcal{A}$ at x.

Theorem 2.10. A grill $\mathcal{G}$ on topological space (X, τ) is $\alpha\delta^{\theta\#}\mathcal{C}$ to apoint x of X if and only if $\mathcal{G} \subseteq sec\ \mathcal{G}\ \alpha\delta^{\theta\#}, x$.

Proof. Let $\mathcal{G}$ be a grill on X, $\alpha\delta^{\theta\#}\mathcal{C}$ to $x \in X$. Then for each $U \in \alpha\delta O\ x$, there exists $G \in \mathcal{G}$ such that $G \subseteq \alpha\delta_{Cl}\ U$, and hence $\alpha\delta_{Cl}\ U \in \mathcal{G}$ for each $U \in \alpha\delta O\ x$. Now, $B \in sec\ \mathcal{G}\ \alpha\delta^{\theta\#}, x$.

$\Rightarrow X - B \notin \mathcal{G}\ \alpha\delta^{\theta\#}, x \Rightarrow$ there exists $U \in \alpha\delta O\ x\ such\ that\ \alpha\delta_{Cl}\ U\ \cap\ X - B\ = \varphi \Rightarrow \alpha\delta_{Cl}\ U\ \subseteq B, where\ U \in \alpha\delta O\ x\ \Rightarrow B \in G.$

Conversely, let if possible, $\mathcal{G}$ not a $\alpha\delta^{\theta\#}\ \mathcal{C}$ to x. Then for some $U \in \alpha\delta O\ x$, $\alpha\delta_{Cl}\ U\ \notin \mathcal{G}$ and hence $\alpha\delta_{Cl}\ U\ \notin sec\ \mathcal{G}\ \alpha\delta^{\theta\#}, x$. Thus for some $A \in \mathcal{G}\ \alpha\delta^{\theta\#}, x$, $A \cap \alpha\delta_{Cl} U = \varphi$. But $A \in \mathcal{G}\ \alpha\delta^{\theta\#}, x$

$\Rightarrow x \in \alpha\delta^{\theta\#}cl(A)$.

$\Rightarrow \alpha\delta_{Cl} U \cap A \neq \varphi$. which is a contradiction.

Theorem 2.11. A grill $\mathcal{G}$ on a topological space (X, τ), $\alpha\delta^{\theta\#}\mathcal{C}$ to a point x of X, τ if and only if $sec\ \mathcal{G}\ \alpha\delta^{\theta\#}, x \subseteq \mathcal{G}$.

Proof. Let $\mathcal{G}$ be a grill on a topological space (X, τ), $\alpha\delta^{\theta\#}\mathcal{C}$ to a point $x \in X$. Then for each $U \in \alpha\delta O\ x$ there exists $G \in \mathcal{G}$ such that $G \subseteq \alpha\delta_{Cl}\ U$, and hence $\alpha\delta_{Cl}\ U\ \in G$ for each $U \in \alpha\delta O\ x$. Now, $B \in sec\ \mathcal{G}\ \alpha\delta^{\theta\#}, x \Rightarrow X \setminus B \notin \mathcal{G}\ \alpha\delta^{\theta\#}$, $x \Rightarrow x \notin\ \alpha\delta\theta ClU \Rightarrow$ there exists $U \in \alpha\delta Ox$ such that $\alpha\delta ClU \cap X \setminus B) = \varphi \Rightarrow \alpha\delta ClU \subseteq$ B, where $U \in \alpha\delta Ox \Rightarrow$ B ∈ G. Conversely, let if possible, $\mathcal{G}$ not to $\alpha\delta^{\theta\#}\mathcal{C}$ to x. Then for some $U\ \in\ \alpha\delta O\ x$, $\alpha\delta\theta_{Cl}\ U\ \notin\ \mathcal{G}$ and hence $\alpha\delta\theta_{Cl}\ U\ \notin\ sec\ \mathcal{G}\ \alpha\delta^{\theta\#}, x$. Thus for some $A\ \in\ \mathcal{G}\ \alpha\delta^{\theta\#}, x$, $A \cap \alpha\delta_{Cl}\ U\ = \varphi$. But $A \in \mathcal{G}\ \alpha\delta^{\theta\#}, x \Rightarrow x \in \alpha\delta\theta_{Cl} A \Rightarrow \alpha\delta_{Cl} U \cap$ U $\neq \varphi$.

3. $\alpha\delta$-CLOSEDNESS AND GRILLS

Definition 3.1 A non empty subset A of a topological space X is called $\alpha\delta$-closed relative to X if for every cover $\mathcal{U}$ of A by $\alpha\delta$-open sets of X, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subseteq \cup\{\alpha\delta_{Cl}\ U : U \in \mathcal{U}_0\}$. If, in addition, $A = X$, then $X$is called a $\alpha\delta$-closed space.

Theorem 3.2. For a topological space$X$, the following statements are equivalent:

(a) X is $\alpha\delta$-closed;

(b) Every maximal filter base $\alpha\delta^{\theta\#}\mathcal{C}$ to some point of X;

(c) Every filter base $\alpha\delta^{\theta\#}$-adhere to some point of X;

(d) For every family V_α: $\alpha \in I$ of $\alpha\delta$- closed sets that $\cap\ V_i$: $i \in I = \varphi$, there exists a finite subset I_0 of I such that $\cap\{\alpha\delta_{int} V_i : i \in I_0\}$.

Proof. $a \Rightarrow b$: Let $\mathcal{F}$ be a maximal filter base on X. Suppose that $\mathcal{F}$ does not $\alpha\delta$-converge to any point of X. Since $\mathcal{F}$ is maximal, $\mathcal{F}$ does not $\alpha\delta$-θ-accumulate at any point of X. For each $x \in X$, there exist $F_x \in \mathcal{F}$ and $V_X \in \alpha\delta O(X, x)$ such that $\alpha\delta_{cl}\ V_X \cap F_x = \varphi$. The family$\{V_X : x \in X\}$ is a cover of X by $\alpha\delta$-open sets of X. By (a), there exists a finite number of points $x_1, x_2, x_3 \ldots x_n$ of X such that $X = \cup\{\alpha\delta_{cl}\ V_{x_i} : i = 1,2,\ldots,n\}$. Since $\mathcal{F}$ is a filter base on X, there exists $F_0 \in \mathcal{F}$ such that $F_0 \subseteq \cap\ F_{x_i} : i = 1,2,\ldots,n$ .Therefore, we obtain $F_0 = \varphi$. This is a contradiction.

$b \Rightarrow c$: Let $\mathcal{F}$ be any filter base on X. Then, there exists a maximal filter base $\mathcal{F}_0$such that $\mathcal{F} \subseteq \mathcal{F}_0$. By (b), $\mathcal{F}_0 \alpha\delta$-$\theta$-converges to some point $x \in X$. For every $F \in \mathcal{F}$and every $V \in \alpha\delta O(X, x)$, there exists $F_0 \in \mathcal{F}_0$such that $F_0 \subseteq \alpha\delta_{cl}\ V$; hence $\varphi \neq F_0 \cap F \subseteq \alpha\delta_{cl} V \cap F$. This shows that $\mathcal{F}$ $\alpha\delta$-θ-accumulates at x.

$(c) \Rightarrow (d)$:Let V_α: $\alpha \in I$ be any family of $\alpha\delta$-closed subsets of X such that $\cap\ V_\alpha$: $\alpha \in I = \varphi$. Let $\Gamma(I)$ denote the ideal of all finite subsets of A. Assume that $\cap\{\alpha\delta_{Int}(V_\alpha) : \alpha \in I\} = \varphi$ for every $I \in \Gamma(I)$. Then, the family $\mathcal{F} = \bigcap_{\alpha\in I} \alpha\delta_{Int}\ V_\alpha : I \in \Gamma\ I$ is a filter base on X. By (c), $\mathcal{F}$ $\alpha\delta$-θ-accumulates at some point $x \in X$. Since $X\backslash V_\alpha : \alpha \in I$ is a cover of X, $x \in X\backslash V_{\alpha 0}$ for some $V_{\alpha 0} \in I$. Therefore, we obtain $X\backslash V_{\alpha 0} \in \alpha\delta O(X, x)$, $\alpha\delta_{Int}(V_{\alpha 0}) \in \mathcal{F}$ and $\alpha\delta_{Cl}\ V_{\alpha 0} \cap \alpha\delta_{Int} V_{\alpha 0} = \varphi$, which is a contradiction.

$(d) \Rightarrow (a)$: Let V_α: $\alpha \in I$ be a cover of X by $\alpha\delta$-open sets. Then $\{X\backslash V_\alpha : \alpha \in I\}$ is a family of $\alpha\delta$-closed subsets of X such that $\cap\ X\backslash V_\alpha$: $\alpha \in I = \varphi$. By(d), there exists a finite subset I_0 of I such that $\cap\{\alpha\delta_{Int}(X\backslash V_\alpha) : \alpha \in I_0\} = \varphi$ hence $X = \cup\{\alpha\delta_{Cl}(V_\alpha) : \alpha \in I_0\}$. This shows that X is $\alpha\delta$-closed.

Theorem 3.3.A topological space X is $\alpha\delta$-closed if and only if every grill on X is $\alpha\delta^{\theta\#}$-convergent in X.

Proof Let $\mathcal{G}$ be any grill on a $\alpha\delta$-closed space X. Then by Theorem 2.6, $sec\ \mathcal{G}$ is a filter on X. Let $B \in sec\ \mathcal{G}$, then $X\backslash B \notin \mathcal{G}$ and hence $B \in \mathcal{G}$(as $\mathcal{G}$is a grill). Thus $sec\ \mathcal{G} \subseteq \mathcal{G}$. Then by Theorem 2.6$(b)$, there exists an ultrafilter $\mathcal{U}$ on X such that $sec\ \mathcal{G} \subseteq \mathcal{U} \subseteq \mathcal{G}$. Now as X is $\alpha\delta$-closed, in view of Theorem 3.2, the ultrafilter $\mathcal{U}$is$\alpha\ \ \delta^{\theta\#}$-convergent to some point $x \in X$. Then for each $U \in \alpha\delta O(X, x)$, there exists $F \in \mathcal{U}$ such that $F \subseteq \alpha\delta_{Cl}(U)$. Consequently, $\alpha\delta_{Cl}(U) \in \mathcal{U} \subseteq \mathcal{G}$, That is $\alpha\delta_{Cl}(U) \in \mathcal{G}$, for each $U \in \alpha\delta O(X, x)$. Hence $\mathcal{G}$ is $\alpha\delta^{\theta\#}$-convergent to x. Conversely, let every grill on X be $\alpha\delta^{\theta\#}$-convergent to some point of X. By virtue of Theorem 3.2 it is enough to show that every ultrafilter on $X$is $\alpha\ \delta^{\theta\#}$-converges in X, which is immediate from the fact that an ultrafilter on $X$is also a grill on X.

Theorem 3.4. A topological space $X$is $\alpha\delta$-closed relative to Xif and only if every grill $\mathcal{G}$ on X with $A \in \mathcal{G}$, $\alpha\delta^{\theta\#}$-converges to a point in A.

Proof. Let A be $\alpha\delta$-closed relative to $X$and $\mathcal{G}$ a grill on X satisfying $A \in \mathcal{G}$ such that $\mathcal{G}$ does not $\alpha\delta^{\theta\#}$-converge to any $a \in A$. Then to each $a \in A$, there corresponds some $U_a \in \alpha\delta O\ X, a$ such that $\alpha\delta_{Cl}(U_a) \notin \mathcal{G}$. Now $U_\alpha : \alpha \in A$ is a cover of A by $\alpha\delta$-open sets of X . Then $A \subseteq {}^{n}_{i=1} \alpha\delta_{Cl} U_{a_i} = U$(say) for some positive integer n. Since $\mathcal{G}$ is a grill, $U \notin \mathcal{G}$; hence $A \notin \mathcal{G}$, which is a contradiction.

Conversely, let$A$ be not $\alpha\delta$-closed relative to X. Then for some cover $\mathcal{U} = U_\alpha : \alpha \in A$ of A by $\alpha\delta$-open sets of X, $\mathcal{F} = \{A\backslash \ _{\alpha\in I_0} \alpha\delta_{Cl} U_\alpha : I_0\ \textit{is finite subset of } I\}$ is a filterbase on X. Then the family $\mathcal{F}$can be extended to an ultrafilter $\mathcal{F}^*$on X. Then $\mathcal{F}^*$is a grill on X with $A \in \mathcal{F}^*$(as each F of $\mathcal{F}$is a subset of A). Now for each $x \in A$, there must exists $\beta \in I$ such that $x \in U_\beta$, as U is a cover of A. Then for any $G \in \mathcal{F}^*$, $G \cap (A\backslash \alpha\delta_{Cl}(U_\beta) \neq \varphi$, so that $G \supset \alpha\delta_{Cl}(U_\beta)$for all $G \in \mathcal{G}$. Hence $\mathcal{F}^*$, cannot $\alpha\delta^{\theta\#}$-converge to any point of A. The contradiction proves the desired result.

REFERENCES

Choquet, G., (1947). Sur les notions de Filtre et grille, Comptes Rendus *Acad. Sci. Paris* **224**: 171-173.

Devi, R., V. Kokilavani and P. Basker, (2012). On Strongly-αδ-Super-Irresolute Functions In Topological Spaces. *Int. J. Comp. Appl.,* **40**(17).

Kokilavani, V. and P. Basker, (2012d). On $\mathcal{F}^{-\alpha\delta}$ Continuous Multifunctions. *Int. J. Comp. Appl.,* **41**(2).

Roy, B. and M.N. Mukherjee, (2009). Concerning topologies induces by principal grills, *An. Stiint. Univ. AL. I. Cuza Iasi. Mat.,* **55**(2): 285-294.