TWO-OUT DEGREE EQUITABLE DOMINATION IN THE MIDDLE, CENTRAL AND THE LINE GRAPHS OF $P_n$, $C_n$ AND $K_{1,n}$

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ABSTRACT

Let $G=(V,E)$ be a simple, finite, connected and undirected graph. A dominating set $D$ of $G$ is said to be two-out degree equitable dominating set if for any two vertices $u, v \in D$ such that $|od_D(u) - od_D(v)| \leq 2$, where $od_D(u) = \left| N(v) \cap (V - D) \right|$. The minimum cardinality of two-out degree equitable dominating set is called two-out degree equitable domination number and it is denoted by $\gamma_{2oe}(G)$. In this paper, we introduced the two-out degree equitable domination numbers in the middle, central and the line graphs of the path $P_n$, cycle $C_n$, and star $K_{1,n}$ graphs.

Keywords: Two-out degree equitable domination number, Middle graph, Central graph, Line graph, Path $P_n$, Cycle $C_n$, and Star graph $K_{1,n}$.

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1. INTRODUCTION

The concept of domination was first studied by Ore and Berge (1962). A non-empty set $D \subseteq V$ is said to be a dominating set of $G$ if every vertex in $V - D$ is adjacent to at least one vertex in $D$. The minimum cardinality of the minimal dominating set $D$ is called the domination number and it is denoted by $\gamma(G)$.

An equitable domination has interesting application in the context of social network. In a network, nodes with nearly equal capacity may interact with each other in a better way. In society, persons with nearly equal status, tend to be friendly. Ali Sahal and V. Mathad (Sahal, 2013) introduced the concept of two out degree equitable domination in graphs. In this paper, we investigated the two out degree equitable domination number in the middle and the central graphs of $P_n$, $C_n$ and $K_{1,n}$ graphs.

Definition 1.1 (8)

The middle graph of a connected graph $G$ denoted by $M(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if

(i) They are adjacent edges of $G$ (or)

(ii) One is a vertex of $G$ and the other is an edge incident with it.

Definition 1.2 (8)

For a given graph $G=(V,E)$ of order $n$, the central graph $C(G)$ is obtained, by subdividing each edge in $E$ exactly once and joining all the non adjacent vertices of $G$. The central graph $C(G)$ of a graph $G$ is an example of a split graph, where a split graph is a graph whose vertex set $V$ can be partitioned into two sets, $V_1$ and $V_2$, where each pair of vertices in $V_1$ are adjacent, and no two vertices in $V_2$ are adjacent.

Definition 1.3 (7)

A dominating set $D$ in a graph $G$ is called a two-out degree equitable dominating set if for any two vertices $u, v \in D$ such that $|od_D(u) - od_D(v)| \leq 2$, where $od_D(u) = \left| N(v) \cap (V - D) \right|$. The minimum cardinality of a two-out degree equitable dominating set is called the two-out degree equitable domination number of $G$ and is denoted by $\gamma_{2oe}(G)$.

In the consequent section, we obtained the two-out degree equitable domination number $\gamma_{2oe}(G)$ of the middle graph of $P_n$, $C_n$ and $K_{1,n}$ graphs.

Definition 1.4 (9)

Line graph $L(G)$ of a graph $G$ is defined with the vertex set $E(G)$, in which two vertices are adjacent if and only if the corresponding edges are adjacent in $G$.

2. Two-out degree Equitable domination in the Middle graphs of $P_n$, $C_n$ and $K_{1,n}$

Example 2.1

Let $G$ be the middle graph as in the figure. We obtained the two-out degree equitable domination number.

*Correspondence: P. Rajakumari, P.G. and Research Department of Mathematics, D.K.M.College for Women (Autonomous), Vellore-632001, Tamilnadu, India. E.mail: rajakumari0990@gmail.com
Let $v_i, v_{i+1}, v_{i+2}, v_{i+3}$ be the path of length $n$ and $u_i u_{i+1} = v_i$. By the definition of middle graph, $M(P_n)$ has the vertex set $V = \{(v_{i+1}, i+1), (v_i, i), (v_{i+3}, i+3) | 0 \leq i \leq n\}$ and each $v_i$ is adjacent to $v_{i+1}$ and $v_i$ is adjacent to $u_{i+1}$. The vertices $u_i, v_i, v_{i+1}, v_{i+2}, v_{i+3}$ of $M(P_n)$ induces a path of length $4k$.

Let $D = \{(v, v_{i+1}, i+2), (v_{i+1}, v_{i+2}, v_{i+3}) | 0 \leq i \leq n\}$ be a dominating set of $M(P_n)$ and $V - D = \{(v_i, v_{i+1}, i=1), (v_{i-1}, v_{i+1}, v_{i+3}/i = 2), ... \} \cup (u_i/1 \leq i \leq n)$. Then $|\{v_{i+1}, v_i + 1, i=1\} \cup (u_i/1 \leq i \leq n)| = 3$ or $4$.

Then $|od_D(v_i) - od_D(v_{i+1})| \leq 2$, for any $v_i v_{i+1} \in D$. Therefore $D$ is the minimum two-out degree equitable dominating set. Hence, $\gamma_{2oe} M(P_n) = \frac{n}{3}$ where $n \geq 5$.

**Theorem 2.3**

For any Cycle $C_n$, $\gamma_{2oe} M(C_n) = \frac{n}{3} + 1$ where $n \geq 4$.

**Proof**

Let $V C_n = \{u_1, u_2, ..., u_n\}$ and $E C_n = \{v_1, v_2, ..., v_n\}$, where $v_i = u_{i+1}, i \leq i \leq n-1$, $v_n = u_1$. By the definition of middle graph, $M(C_n)$ has the vertex set $V = \cup E C_n$ in which each $v_1$ is adjacent to $v_{i+1}$ and each $v_i$ is adjacent to $u_{i+1}$. In $M(C_n)$, $u_1, v_2, v_3$ induces a cycle of length $2n$. That is $|V M C_n| = 2n$ and $|E M C_n| = 3n$.

Let $D = \{(v_1, v_{i+2}/i = 1), (v_{i-1}, v_{i+1}, v_{i+3}/i = 2), ... \}$ be a dominating set of $M C_n$ and $V - D = \{(v_i, v_{i+2}/i = 1), (v_{i-1}, v_{i+1}/i = 2), ... \} \cup \{u_i/1 \leq i \leq n\}$.

Now $u_i \in D$ then $od_D v_i = |N v_i \cap V - D|$ for $i = 1, od_D v_1 = |N v_1 \cap V - D| = \frac{n}{3}$.

Thus $|\{v_{i+1}, v_i + 1, i=1\} \cup (u_i/1 \leq i \leq n)| = 3$ or $4$.

Then $|od_D(v_i) - od_D(v_{i+1})| \leq 2$, for any $v_i v_{i+1} \in D$. Therefore $D$ is the minimum two-out degree equitable dominating set. Hence, $\gamma_{2oe} M(C_n) = \frac{n}{3} + 1$ where $n \geq 4$.

**Theorem 2.4**

For any Star graph $K_{1,n}$, $\gamma_{2oe} M(K_{1,n}) = n$.

**Proof**

Let $V K_{1,n} = \{u, v_1, v_2, ..., v_n\}$ and $E(K_{1,n}) = \{v_i, v_{i+1}, v_{i+2}\}$, By the definition of middle graph, we have $V M K_{1,n} = u \cup v_i/1 \leq i \leq n \cup u_i/1 \leq i \leq n$ in which the vertices $v_1, v_2, ..., v_n, u_1, u_2, ..., u_n$...
induces clique of order \( n + 1 \). Let \( D = \{ v_i / 1 \leq i \leq n \} \) be the dominating set of \( M(K_{1,n}) \) and \( V - D = (u, u_i) / 1 \leq i \leq n \).

Now \( v_i \in D \) then \( od_D v_i = |N(v_i) \cap V - D| \) for \( i = 1 \), \( od_D v_i = |N(v_i) \cap V - D| \)

\[
= |v_{i+1}, \ldots, v_n \cup u_1, u_2| \leq n \cap u, u_i
\]

\[
= |u, u_i| \text{ since } i = 1
\]

\[
= 2
\]

Then \( |od_D(v_i) - od_D(v_j)| \leq 2 \), for any \( v_i, v_j \in D \). Therefore D is the minimum two - out degree equitable dominating set, Hence, \( \gamma_{2oe} M(K_{1,n}) = n \).

3. Two-out degree Equitable domination in the Central graphs of \( P_n \), \( C_n \) and \( K_{1,n} \).

In this section, we obtained the two-out degree equitable domination number \( \gamma_{2oe}(G) \) of the central graphs of the path \( P_n \), cycle \( C_n \) and the star graph \( K_{1,n} \).

Example 3.1.

Let \( G \) be the central graph as in the figure. we obtained the two-out degree equitable domination number.

Central graph of \( P_5 \)

Consider the set \( D = \{ v_1, u_2, v_3, u_4 \} \). It is a dominating set and \( V - D = \{ v_2, v_3, v_6, v_5, v_1 \} \)

\[
od_D v_i = |N(v_i) \cap V - D|
\]

\[
= |\{ u_1, v_3, v_6, v_5 \} \cap \{ v_2, v_3, v_6, v_5, v_1 \}|
\]

\[
= 4
\]

\[
od_D u_2 = |N(u_2) \cap V - D|
\]

\[
= 2
\]

Similarly, \( od_D u_3 = od_D u_4 = 2 \)

From the above, any two vertices \( u, v \in D \) are such that \( |od_D u - od_D(v)| \leq 2 \).

Therefore \( \{ v_1, u_2, v_3, u_4 \} \) is the minimum two-out degree equitable dominating set with the minimum cardinality 4. Hence, \( \gamma_{2oe} C(P_5) = 4 \).

Theorem 3.2.

For any Path \( P_n \), \( \gamma_{2oe} C(P_n) = n - 1 \) for \( n \leq 5 \).

Proof

Let \( P_n \) be the path of length \( (n - 1) \) with vertices \( v_1, v_2, \ldots, v_n \). By the definition of central graph, the non-adjacent vertices \( v_i \) and \( v_j \) of \( P_n \) are adjacent in \( C P_n \).

Therefore, \( V(C P_n) = \{ v_i / 1 \leq i \leq n \} \) union \( \{ u_1 / 1 \leq i \leq n - 1 \} \) and

\[
E(C P_n) = \{ e / 1 \leq i \leq n - 1 \} \cup \{ e' / 1 \leq i \leq n - 1 \} \cup e'' : 1 \leq i \leq n - 2, i + 2 \leq j \leq n.
\]

Let \( \{ v_i \} \cup \{ u_i / 2 \leq i \leq n - 1 \} \) will be the dominating set.

Let

\[
D = (v_i, u_{i+2}), (v_j, u_{j+1}), (v_k, u_{k-1}, u_i, u_{i+1}) \), where \( 1 \leq i \leq 3 \) be the dominating set of \( C P_n \) and

\[
V - D = u_1, v_{i+2}, u_{j+1}, v_{j+1}, v_{i+1}, u_1, v_{i+1}, v_{i+2}, u_1, v_{i+2}, v_{i+1}, v_{i+2}
\]

where \( 1 \leq i \leq 3 \) and

Now \( v_i \in D \) then \( od_D v_i = |N(v_i) \cap V - D|
\]

\[
= |(u_1, v_{i+2}), (u_{i+1}, v_{i+2}, v_{i+1}), (u_{i+1}, v_{i+1}, v_{i+2})|
\]

\[
\cap \{ u_1, v_{i+1}, v_{i+2}, (u_{i+1}, v_{i+1}, v_{i+2}), (u_{i+1}, v_{i+1}, v_{i+2}) \}
\]

\[
= |(u_1, v_{i+2}), (u_{i+1}, v_{i+1}, v_{i+2})|
\]

\[
= n - 1
\]

Now \( u_i \in D \) then \( od_D u_i = |N(u_i) \cap V - D|
\]

\[
= 2
\]

Then \( |od_D(v_i) - od_D(u_i)| \leq 2 \) for any \( v_i, u_i \in D \). Therefore D is the minimum two - out degree equitable dominating set. Hence, \( \gamma_{2oe} C(P_n) = n - 1 \) for \( n \leq 5 \).

Theorem 3.3.

For any Cycle \( C_n \), \( \gamma_{2oe} C(C_n) = n - 1 \) for \( n \leq 5 \).

Proof

Let \( C_n \) be any cycle of length \( n \) and let \( V C_n = \{ v_1, v_2, \ldots, v_n \} \) and \( E C_n = e_1, e_2, \ldots, e_n \).

By the definition of central graph \( C(C_n) \) has the vertex set \( V C_n \cup \{ u_i : 1 \leq i \leq n \} \) where \( u_i \) is a vertex of subdivision of the edge \( v_i, v_{i+1} (i \leq i \leq n - 1) \) and \( u_n \) is a vertex of subdivision of the edge \( v_1, v_n \).

Let

\[
D = (v_i, u_{i+1}), (v_j, u_{j+1}), (v_k, u_{k-1}, u_i, u_{i+1}) \), where \( 1 \leq i \leq 3 \) be a dominating set of \( C C_n \) and \( V - D = \{ v_{i+1}, v_{i+2}, u_{i+2}, (v_{i+1}, v_{i+2}, u_{i+2}), (v_{i+1}, v_{i+2}, u_{i+2}) \} \), where \( 1 \leq i \leq 3 \).
Now \( v_i \in D \) then \( od_D v_i = |N v_i \cap V - D| \)
\[
= |(u_i+2, (u_{i-2}, u_{i+2}, v_i, v_i), (u_{i-3}, u_{i+2}, v_i, v_i, v_i))| \\
= |(u_i+2), (u_{i-1}, u_{i+2}, v_i, v_i), (u_{i-2}, u_{i+2}, v_i, v_i)| \\
= n - 2 \text{ or } n - 1
\]
Similarly, \( od_D u_i = od_D u_{i+1} = 2 \)

Then \( |od_D(v_i) - od_D(u_{i+1})| \leq 2 \), for any \( v_i \in D \). Therefore D is the minimum two - out degree equitable dominating set. Hence, \( \gamma_{20e} C_n(n) = n - 1 \) for \( n \leq 5 \).

**Theorem 3.4.**
For any Star graph \( K_{1,n} \), \( \gamma_{20e} C(K_{1,n}) = 2 \).

**Proof**

Let \( V = \{v_1, v_2, ... v_n \} \) and \( C(K_{1,n}) \) be the dominating set of \( C(K_{1,n}) \) and \( V = (v_1, v_1, v_2, ..., v_n) \). Now \( od_D v_i = |N v_i \cap V - D| \)
\[
= |{(v_1, v_1, v_2, ..., v_n) \cap (v_1, v_1, v_2, ..., v_n)}| \\
= n
\]
Similarly, \( od_D v_{i+1} = n \)

Then \( |od_D(v_i) - od_D(v_{i+1})| \leq 2 \), for any \( v_i, v_{i+1} \in D \). Therefore D is the minimum two - out degree equitable dominating set. Hence, \( \gamma_{20e} C(K_{1,n}) = 2 \).

**4. Two-out degree Equitable domination in the Line graphs of \( P_n \), \( C_n \) and \( K_{1,n} \)**

In this section, we obtained the two-out degree equitable domination number \( \gamma_{20e} \) of the line graphs of the path \( P_n \), cycle \( C_n \) and the star graph \( K_{1,n} \).

**Theorem 4.1.**
For any Path \( P_n \), \( \gamma_{20e} L(P_n) = n - 3 \).

**Proof**

Let \( P_n \) has \( n \) vertices -1 edges. Let \( V = \{v_1, v_2, ..., v_{n-1} \} \) be the vertices of \( L(P_n) \) and let \( E = \{u_1, u_2, ..., u_{n-2} \} \) be the edges of \( L(P_n) \).

Since the degree of any vertex in \( L(P_n) \) is 2 except the initial and terminal vertices.
Let us consider \( D = \{v_1, v_2, ..., v_{n-3} \} \) be a dominating set of \( L(P_n) \) and \( V = (v_1, v_2, ..., v_{n-3}) \)

Now \( v_i \in D \) then \( od_D v_i = |N v_i \cap V - D| \)
\[
od_D v_{i+1} = |N v_{i+1} \cap V - D| = 0
\]
Similarly, \( od_D v_{n-3} = 1 \). Hence for every \( v_i, v_{i+1} \in D \) then \( |od_D u - od_D(v)| \leq 2 \). So D is the minimum two - out degree equitable dominating set. Then \( \gamma_{20e} L(P_n) = n - 3 \).

**Theorem 4.2.**
For any Cycle \( C_n \), \( \gamma_{20e} L(C_n) = n - 2 \).

**Proof**

Let \( L(C_n) \) have \( n \) vertices and \( n \) edges in which each vertex is of degree 2. That is each vertex dominates two vertices. Let \( V = \{u_1, u_2, ..., u_n \} \) be the vertices of \( L(C_n) \).

Let us consider \( D = \{u_1, u_2, ..., u_{i-1}, u_{i+1}, ..., u_n \} \) be the dominating set of \( L(C_n) \) and \( V = (u_1, u_{i+1}) \).
Now \( od_D(u_1) = 0, j = 1, 2, ..., i - 2, i + 3, ..., n \)
\[
od_D(u_1) = 1 \text{ and } od_D(u_{i+1}) = 1
\]
Then \( |od_D(u_1) - od_D(u_{i+1})| \leq 2 \)

Then D is the minimum two out degree equitable dominating set

So \( \gamma_{20e} L(C_n) = n - 2 \).

**Note:** The line graph of \( C_n \) is \( \gamma_{20e} C_n(n) = n \).

**Theorem 4.3.**
For any Star graph \( K_{1,n} \), \( \gamma_{20e} L(K_{1,n}) = 1 \) for \( n \leq 3 \).

**Proof**

Let \( V = \{v_1, u_1, u_2, ..., u_n \} \) be the vertices of \( K_{1,n} \). Let \( V = \{v, u_1, u_2, ..., u_{n-1} \} \) be the vertices of \( L(K_{1,n}) \). Let us consider \( D = \{v \} \) be a dominating set of \( L(K_{1,n}) \) and \( V = (u_1, u_2) \).

Now \( v \in D \) then \( od_D v = |N v \cap V - D| \)
\[
= |u_1, u_2 \cap (u_1, u_2)| = 2
\]
Then \( |od_D(u) - od_D(v)| \leq 2 \), for any \( u, v \in D \). Therefore D is the minimum two - out degree equitable dominating set. Hence, \( \gamma_{20e} L(K_{1,n}) = 1 \) for \( n \leq 3 \).

**5. CONCLUSION**

In this paper, we introduced two-out degree equitable domination number in the middle, central and line graph of \( P_n \), \( C_n \), and \( K_{1,n} \) graphs. We extend this study on some more special classes of graphs.

**REFERENCES**


