RESEARCH ARTICLE

ON STAR COLOURING OF M(T_{M,N}), M(T_N), M(L_N) AND M(S_N)

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ABSTRACT

A Star coloring of an undirected graph G is a proper vertex coloring of G in which every path on four vertices contains at least three distinct colors. The Star chromatic number of an undirected graph G, denoted by $\chi_s(G)$ is the smallest integer k for which G admits a star coloring with k colors. In this paper, we obtain the exact value of the Star chromatic number of Middle graph of Tadpole graph, Snake graph, Ladder graph and Sunlet graphs denoted by $M[T_{m,n}]$, $M[T_n]$, $M[L_n]$ and $M[S_n]$ respectively.

Keywords: Star coloring, Star chromatic number, Middle graph.

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper we consider the graph G = (V,E) as a undirected, simple, finite and connected graph with no loops. A vertex coloring (4) of a graph is said to be proper coloring if no two adjacent vertices have the same color. In vertex coloring of G, the set of vertices with same color is known as color class. A proper vertex coloring of a graph is said to be star coloring if the induced subgraph of any two color classes is a collection of stars.

In 1973, Branko Grünbaum (5) introduced the concept of star coloring and also he introduce the notion of star chromatic number. In the beginning, he developed a new concept called acyclic coloring, where it is required that every cycle uses at least 3 colors, so the 2 color induced subgraphs are Forests. Later he established the star coloring concept as a special type of acyclic coloring. His works were developed further by Bondy and Hell (4). A star coloring of a graph is a vertex coloring such that the union of any two color classes does not contain a bicolored path of length 3 and the star chromatic number of a graph is a minimum number of colors which are necessary to star color the graph.

In 2004, Guillaume Fertin *et al.* (6) developed the exact value of the star chromatic number of different families of graphs such as cycles, trees, 2-dimensional grids, complete bipartite graphs, and outer planar graphs. Further, the authors studied and gave bounds for the star chromatic number of some graphs. Albertson *et al.* (4) proved that finding the star chromatic number is NPcomplete to find out whether $\chi_s(G) \leq 3$, even when G is a graph that is both planar and bipartite.

They have also proved that, if G is a graph with minimum degreeΔ, the $\chi_s(G) \leq \Delta\Delta - 1 + 2$.

In 1974, The concept of middle graph was introduced by Hamada and Yoshimura (4).

Let G be a graph with vertex set V(G) and edge set E(G). The Middle graph (3) of G, denoted by M(G), is defined as follow. The vertex set of M(G) is V(G) ∪ E(G) and any two vertices x,y in M(G) are adjacent in M(G), if one of the following cases holds: (i) x,y are in E(G) and x,y are adjacent in G. (ii) x is in V(G), y is in E(G) and x,y are adjacent in G.

The (m,n)-Tadpole graph(2) is a graph is obtained by joining a cycle graph C_m to a path graph P_n with a bridge. It is also known as dragon graph. It is denoted by $T_{m,n}$.

A Snake graph (3) is a Eulerian path in the hypercube that has no chords. In other words, any hypercube edge joining snake vertices is a snake edge. It is denoted by T_n.

The Ladder graph (7) is a planar undirected graph with 2n vertices and 3n-2 edges. It is denoted by L_n.

The n-Sunlet graph (1) is the graph on 2n vertices obtained by attaching n pendent edges to the cycle graph C_n. It is denoted by S_n.

2. STAR COLORING OF M ($T_{m.n}$)

Theorem 2.1. For the Tadpole graph $T_{m,n}$, where n>1 and m≥1, the Star chromatic number of Middle graph of Tadpole graph is 5.

(i.e)., $\chi_s[M(T_{m,n})] = 5$, n>1 and m≥1.

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Proof

Let $T_{m,n}$ be the Tadpole graph with joining the cycle C_m and path P_n, with $m \geq 1$ and $n > 1$. Clearly, $V(T_{m,n}) = \{ v_i : 1 \leq i \leq n+m \}$ and $E(T_{m,n}) = \{ e_i : 1 \leq i \leq n+m \}$

By the definition of Middle graph, subdividing each edge of $T_{m,n}$ exactly once and joining all the new vertices of adjacent edges of $T_{m,n}$. The vertex set of Middle graph of $T_{m,n}$ is, $V[M(T_{m,n})] = \{ v_i : 1 \leq i \leq n+m \} \cup \{ u_i : 1 \leq i \leq n+m \}$

Let us consider the vertices $\{ v_i : 1 \leq i \leq m+n \}$ and $\{ u_i : 1 \leq i \leq m+n \}$ in counter clockwise direction. It is clear that, the vertices $\{ v_1 \cup u_1 \cup u_m \cup u_{m+1} : m \geq 3 \}$ induce a clique of order 4. Therefore $\chi_s[M(T_{m,n})] \geq 5$.

Consider the color class $C = \{ c_i : 1 \leq i \leq 5 \}$. Assign a proper star coloring as follows

$C(v_i) = c_1$, for $1 \leq i \leq n$

$$C(u_i) = \begin{cases} c_2, \text{ for } i = 1 \\ c_3, \text{ for } i \equiv 0 \pmod 2 \\ c_4, \text{ for } i \equiv 1 \pmod 4 \\ c_5, \text{ otherwise} \end{cases}$$

From the above coloring procedure, it is clear that every path on 4 vertices contains at least 3 distinct colors which satisfies the definition of star coloring. Therefore $\chi_s[M(T_{m,n})] \leq 5$. Hence, $\chi_s[M(T_{m,n})] = 5$.

2.1 Structural Properties of Middle Graph of Tadpole Graph

- Number of vertices in $M(T_{m,n}) = 2(n+m)$
- Number of edges in $M(T_{m,n}) = 3(n+m)+1$
- $\Delta[M(T_{m,n})] = 5$
- $\delta[M(T_{m,n})] = 1$

3. STAR COLORING OF M (T_n)

Theorem 3.1 For $n \geq 2$, the Star chromatic number of Middle graph of Snake graph is,

$\chi_s[M(T_n)] = 6$

Proof

By the definition of Triangular snake graph, it is a connected graph all of whose blocks are triangle. Clearly, $V(T_n) = \{ v_i \cup u_j : 1 \leq i \leq n+1, 1 \leq j \leq n \}$ and $E(T_n) = \{ e_i : 1 \leq i \leq 3n \}$

Consider $M(T_n)$, the vertex set of Middle graph of T_n is,

$V[M(T_n)] = \{ v_i \cup u_j \cup w_i \cup x_l : 1 \leq i \leq n+1, 1 \leq j \leq n, 1 \leq l \leq 2n \}$

Clearly, the vertices $\{ v_i \cup w_j \cup w_{j+1} \cup x_{2k} \cup x_{2k+1} : 2 \leq i \leq n-1, 1 \leq j \leq n, 1 \leq k \leq 2n-1 \}$ induce a clique of order 5. Therefore $\chi_s[M(T_n)] \geq 6$.

Consider the color class $C = \{ c_i : 1 \leq i \leq 6 \}$.

Now, assign a proper star coloring as follows

$C(v_i)$ and $C(u_j) = c_1$, for $1 \leq i \leq n+1, 1 \leq j \leq n$

$$C(w_j) = \begin{cases} c_3, \text{ for } j = 3 \text{ and } j \equiv 3 \pmod 4 \\ c_4, \text{ for } j = 1 \text{ and } j \equiv 1 \pmod 4 \\ c_5, \text{ for } j \equiv 0 \bmod 4 \\ c_6, \text{ for } j = 2 \text{ and } j \equiv 2 \pmod 4 \end{cases}$$

$$C(x_k) = \begin{cases} c_2, \text{ for } k \equiv 1 \pmod 2 \\ c_3, \text{ for } k = 2 \text{ and } k \equiv 2 \pmod 8 \\ c_4, \text{ for } k \equiv 2 \bmod 4 \\ c_5, \text{ otherwise} \end{cases}$$

An easy check shows that, no path on 4 vertices is bicoloured which satisfies the definition of Star coloring, Therefore, $\chi_s[M(T_n)] \leq 6$.

Hence, $\chi_s[M(T_n)] = 6$.

3.1 Structural Properties of Middle Graph of Snake Graph

- Number of vertices in $M(T_n) = 5n+1$
- Number of edges in $M(T_n) = 13n-4$
- $\Delta[M(T_n)] = 8$
- $\delta[M(T_n)] = 2$

4. STAR COLORING OF M (L_n)

Theorem 4.1 For $n \geq 5$, the Star chromatic number of Middle graph of Ladder graph is 6.

(i.e.), $\chi_s[M(L_n)] = 6$, $n \geq 5$.

Proof

Let L_n be the Ladder graph with $2n$ vertices and $3n-2$ edges. Let $\{ v_1, v_2, v_3, \dots v_n \} \cup \{ v'_1, v'_2, v'_3, \dots v'_n \}$ be the vertices of Ladder graph L_n. (i.e.), $V(L_n) = \{ v_i \cup v'_i : 1 \leq i \leq n \}$ and $E(L_n) = \{ e_j : 1 \leq j \leq 3n-2 \}$

Consider $M(L_n)$, the vertex set of Middle graph of L_n is,

$V[M(L_n)] = \{ v_i \cup v' \cup u_j \cup u' \cup w_i : 1 \leq i \leq n, 1 \leq j \leq n-1 \}$

Consider the color class $C = \{ c_i : 1 \leq i \leq 6 \}$.

Now, assign a proper star coloring as follows

$C(v_i)$ and $C(v'_j)$ = c1, for $1 \le i \le n$,

$$C(u_j) \text{ and } C(u'_j) = \begin{cases} c_2, \text{ for } j = 1 \text{ and } j \equiv 1 (mod\ 2) \\ c_3, \text{ for } j = 2 \text{ and } j \equiv 2 (mod\ 4) \\ c_4, \text{ for } j \equiv 0 (mod\ 4) \end{cases}$$

$$C(w_i) = \begin{cases} c_5, \text{ for } i = 1 \text{ and } i \equiv 1 (mod\ 2) \\ c_6, \text{ for } i \equiv 0 (mod\ 2) \end{cases}$$

From the above coloring procedure, it is clear that no path on 4 vertices is bicolored.

Hence, $\chi_s[M(L_n)] = 6$.

4.1 Structural Properties of Middle Graph of Ladder Graph

- Number of vertices in $M(L_n)$ = 5n-2

- Number of edges in $M(L_n)$ = 12(n-1)

- $\Delta[M(L_n)]$ = 6

- $\delta[M(L_n)]$ = 2

5. STAR COLORING OF M (S_n)

Theorem 5.1 For $n \ge 3$ and $n \ne 5$, the Star chromatic number of Middle graph of Sunlet graph is 5.

(i.e)., $\chi_s[M(S_n)] = 5$, $n \ge 3$ and $n \ne 5$.

Proof

The Sunlet graph is the graph on 2n vertices obtained by attaching n pendant edges to a cycle C_n. Clearly, $V(S_n) = \{v_i \cup u_i : 1 \le i \le n \}$ and $E(S_n) = \{e_j : 1 \le j \le 2n\}$

By the definition of Middle graph, each edges of graph is subdivided exactly once by a new vertex and joining all the new vertices of adjacent edges of L_n. The vertex set of middle graph of S_n is, $V[M(S_n)]$ = $\{ v_i \cup u_i \cup w_i \cup x_i : 1 \le i \le n \}$

Let us consider the vertices $\{v_i \cup u_i \cup w_i \cup x_i : 1 \le i \le n\}$ in the counter clockwise direction.

Consider the color class C = $\{ c_i : 1 \le i \le 5 \}$ and assign a proper star coloring as follows

$C(v_i)$ and $C(u_i)$ = c1, for $1 \le i \le n$

$C(x_i)$ = c2, for $1 \le i \le n$

Case (i) : For $i \equiv 0\ mod\ 4$

To admit star coloring, for $1 \le i \le n$ assign the color sequences 3,4,3,5,3,4,3,5....3,4,3,5 to the successive vertices of w_i. An easy check shows that, it requires five minimum colors to color the vertices of $M(S_n)$ to satisfy the definition of star coloring.

Case (ii): For $i \equiv 3 (mod\ 4)$

To admit star coloring , for $1 \le i \le$ n-3 assign the color sequences 3,4,3,5,3,4,3,5,...3,4,3,5 to the successive vertices of w_i and for n-2 $\le$ i $\le$ n assign the colors 3,4,5 to the respective vertices of w_i. It requires five minimum colors to color the vertices of $M(S_n)$ to satisfy the definition of star coloring.

Case (iii): For $i \equiv 5 (mod\ 4)$

To admit star coloring, for i=5,6 assign the colors 4 and 5 to the vertices of w_i respectively, for n-2 $\le$ i $\le$ n assign the colors 3,4,5 to the respective vertices of w_i and for $1 \le i \le$ n-3 assign the color sequence 3,4,3,5,3,4,3,5,...3,4,3,5 to the successive vertices of w_i. An easy check shows that, it requires five minimum colors to color the vertices of $M(S_n)$ to satisfy the definition of star coloring

Case (iv): For $i \equiv 2 (mod\ 4)$

To admit star coloring, for i=5,6 assign the colors 4 and 5 to the vertices of w_i respectively and for $1 \le i \le$ n assign the color sequences 3,4,3,5,3,4,3,5....3,4,3,5 to the successive vertices of wi. It requires five minimum colors to color the vertices of $M(S_n)$ to satisfy the definition of star coloring.

By the above cases, it is clear that, no path on 4 vertices is bicoloured.

Hence, $\chi_s[M(S_n)] = 5$.

5.1 Structural Properties of Middle Graph of Sunlet Graph

- Number of vertices in $M(S_n)$ = 4n

- Number of edges in $M(S_n)$ = 7n

- $\Delta[M(S_n)]$ = 6

- $\delta[M(S_n)]$ = 1

6. CONCLUSION

In this paper, we have found the Star chromatic number for Middle graph of Tadpole graph, Snake graph, Ladder graph and Sunlet graph. Further the results will be extended to obtain the bounds for various graphs.

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