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RESEARCH ARTICLE

ON DYNAMIC COLORING OF WEB GRAPH

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ABSTRACT

Dynamic coloring of a graph G is a proper coloring. The chromatic number of a graph G is the minimum k such that G has a dynamic coloring with k colors. In this paper we investigate the dynamic chromatic number for the Central graph, Middle graph, Total graph and Line graph of Web graph W_n denoted by $C(W_n)$, $M(W_n)$, $T(W_n)$ and $L(W_n)$ respectively.

Keywords: Dynamic coloring, Web graph, Middle graph, Total graph, Central graph and Line graph.

1. INTRODUCTION

Throughout this paper all graphs are finite and simple. The dynamic chromatic number was first introduced by Montgomery (13). A dynamic coloring is defined as a proper coloring in which any multiple degree vertex is adjacent to more than one color class. A dynamic coloring is thus a map c from V to the set of colors such that

- If $uv \in E(G)$, then $c(u) = c(v)$, and
- For each vertex $v \in V(G)$, $|c(N(v))| \geq \min\{2, d(v)\}$

The first condition characterizes proper colorings, the adjacency condition and second condition is double-adjacency condition. The dynamic chromatic number $\chi_d = \chi_d(G)$ is the minimum k for which G has a dynamic k-coloring. The dynamic chromatic number, $\chi_d(G)$, has been investigated in several papers, see, (1, 2, 3, 4, 5, 8, 13, 14). In 2001 Montgomery (13) conjectured that for a regular graph G, $\chi_d(G) - \chi(G) \leq 2$. Akbari *et al.* (2) proved this conjecture for bipartite regular graphs. Some upper bounds for the dynamic chromatic number of graphs have been studied in recent years. In (11, 12), Mohanapriya *et al*. studied δ-dynamic chromatic number of helm and fan graph families. There are many upper bounds and lower bounds for $\chi_d(G)$ in terms of graph parameters. For example, Theorem A [13]. Let G be a graph with maximum degree $\Delta(G)$. Then $\chi_d(G) \leq \Delta(G) + 3$. In this regard, for a graph G with $\Delta(G) \geq 3$, it was proved that $\chi_d(G) \leq \Delta(G) + 1$ [8]. Also, for a regular graph G, it was shown by Alishahi: Theorem (4). [rgb]1,0,0 If G is a r-regular graph, then $\chi_d(G) \leq \chi(G) + 14.06 \log r + 1$.

Another upper bound on $\chi_d(G)$ is $\chi_d(G) \leq 1 + l(G)$, where $l(G)$ the length of a longest path in G (13) is. Theorem C (7). If G is a connected planar graph with $G \neq C5$, then $\chi_d(G) \leq 4$ Alishahi (4) proved that for every graph G with $\chi\ G \geq 4$, $\chi_d(G) \leq \chi(G) + \gamma(G)$, where $\gamma(G)$ is the domination number of a graph. Another upper bound for the dynamic chromatic number of a d −regular graph G in terms of $\chi(G)$ and the independence number of G, $\alpha(G)$, was introduced in (5). In fact, it was proved that $\chi_d(G) \leq \chi(G) + 2\log 2\ \alpha(G) + 3$. In (10), it has been proved that the computational complexity of $\chi_d(G)$ for a 3-regular graph is an NP-complete problem. Furthermore, in (9) it is shown that it is NP-complete to determine whether there exists a 3-dynamic coloring for a claw free graph with the maximum degree 3.

2. PRELIMINARIES

When it is required for an edge $uv = e \in E(G)$ to be represented by a vertex such vertex will be denoted by e'. The line graph (6) of a graph G, denoted by $L(G)$, is the graph in which, all edges $e_i \in E(G)$ are represented by $e_i' \in V(L(G))$ and an edge $e_i'e_j' \in E(L(G))$ if and only if the edges e_i, e_j share a vertex (are incident) in G.

The middle graph of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds:

(i) x, y are in $E(G)$ and x, y are adjacent in G.

(ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

The total graph (6) of G, denoted by has vertex set $V(G) \cup E(G)$, and edges joining all elements of this vertex set which are adjacent or incident in G.

The central graph (15) of a graph G denoted by $C(G)$ is formed by subdividing each

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edge of G by a vertex, and joining each pair of vertices of the original graph which were previously non-adjacent.

3. SOME PROPERTIES OF MIDDLE, TOTAL, CENTRAL AND LINE GRAPHS

MIDDLE GRAPH

- The number of vertices in $M(W_n) = 7n$
- The number of edges in $M(W_n) = 17n$.
- The maximum degree of $M(W_n) = \Delta(M(W_n)) = 8$.
- The minimum degree of $M(W_n)$, $\delta(M(W_n)) = 1$.
- The number of vertices having maximum degree Δ in $M(W_n) = n$.
- The number of vertices having minimum degree δ in $M(W_n) = n$.

TOTAL GRAPH

- The number of vertices in $T(W_n) = 7n$.
- The number of edges in $T(W_n) = 21n$.
- The maximum degree of $T(W_n)$, $\Delta(T(W_n)) = 8$.
- The minimum degree of $T(W_n)$, $\delta(T(W_n)) = 2$.
- The number of vertices having maximum degree Δ in $T(W_n) = 2n$.
- The number of vertices having minimum degree δ in $T(W_n) = n$.

CENTRAL GRAPH

- The number of vertices in $C(W_n) = 7n$.
- The number of edges in $C(W_n) = \frac{3n\ 3n+1}{2} + n$.
- The maximum degree of $C(W_n)$, $\Delta(C(W_n)) = 3n - 1$.
- The minimum degree of $C(W_n)$, $\delta(C(W_n)) = 2$.
- The number of vertices having maximum degree Δ in $C(W_n) = 3n$.
- The number of vertices having minimum degree δ in $C(W_n) = 4n$.

LINE GRAPH

- The number of vertices in $L(W_n) = 4n$.
- The number of edges in $L(W_n) = 9n$.
- The maximum degree of $L(W_n)$, $\Delta(L(W_n)) = 6$.
- The minimum degree of $L(W_n)$, $\delta(L(W_n)) = 3$.
- The number of vertices having maximum degree Δ in $L(W_n) = n$.
- The number of vertices having minimum degree δ in $L(W_n) = n$.

4. MAIN RESULTS

Theorem 4.1 If $n \geq 3$ the dynamic chromatic number of the middle graph of web graph $M(w_n)$, $\chi_d(M(w_n)) = 5$.

Proof

Let $V(Wn) = \{ v_i, u_i, w_i \ for\ 1 \leq i \leq n\}$ and Let $(M(Wn)) = \{ v_i, u_i, w_i \ for\ 1 \leq i \leq n \} \cup \{ p_i, a_i, b_i, c_i \ for\ 1 \leq i \leq n \}$, where p_i, a_i, b_i, c_i are the vertices of $M(W_n)$ corresponding to the edge $v_i v_{i+1}$ of $W_n (1 \leq i \leq n-1)$, $v_i u_i$ of $W_n (1 \leq i \leq n)$, $u_i u_{i+1}$ of $W_n (1 \leq i \leq n-1)$, $u_i w_i$ of $W_n (1 \leq i \leq n)$.

Note that any three consecutive vertices of the path must be colored differently in any dynamic coloring of $M(W_n)$, since the first and third vertices are the only neighbours of second vertex and must be colored differently (by double adjacency conditions) and also differently from the second vertex.

Therefore, $\chi_d(M(W_n)) \geq 5$.

Consider 5-coloring of $M(W_n)$ as dynamic.

Case 1 if n is even

- For $1 \leq i \leq n$, assign the color c_1 to the vertices v_i
- For $1 \leq i \leq n$, assign the color c_1 to the vertices u_i
- For $1 \leq i \leq n$, assign the color c_4 to the vertices w_i
- For $1 \leq i \leq n$, ∀ n is odd, assign the color c_2 to vertices p_i
- For $1 \leq i \leq n$, ∀ n is even, assign the color c_3 to vertices p_i
- For $1 \leq i \leq n$, assign the color c_4 to the vertices a_i
- For $1 \leq i \leq n$, ∀ n is odd, assign the color c_2 to the vertices b_i
- For $1 \leq i \leq n$, ∀ n is even, assign the color c_3 to the vertices b_i
- For $1 \leq i \leq n$, assign the color c_5 to the vertices c_i

Case 2 if n is odd

- For $1 \leq i \leq n$, assign the color c_1 to the vertices v_i

• For $1 \leq i \leq n$, assign the color c_1 to the vertices u_i

• For $1 \leq i \leq n$, assign the color c_4 to the vertices w_i

• For $1 \leq i \leq n-1$, $\forall\, n$ is odd, assign the color c_2 to vertices p_i

• For $1 \leq i \leq n-1$, $\forall\, n$ is even, assign the color c_3 to vertices p_i and assign the color c_4 to vertex p_n

• For $2 \leq i \leq n-1$, assign the color c_4 to the vertices a_i and assign c_3 to a_1 and c_2 to vertex a_n

• For $1 \leq i \leq n$ – 1, $\forall\, n$ is odd, assign the color c_2 to the vertices b_i

• For $1 \leq i \leq n$ – 1, $\forall\, n$ is even, assign the color c_3 to the vertices b_i and assign c_4 to vertex b_n

• For $1 \leq i \leq n$, assign the color c_5 to the vertices c_i

From the cases above, it follows that $\chi_d\,(M(W_n)) \leq 5$.

Hence, $\chi_d\,(M(W_n)) = 5$, $\forall\, n \geq 3$. An easy check shows that this is minimum dynamic 5-coloring.

Theorem 4.2. If $n \geq 3$ the dynamic chromatic number of the total graph of web graph $T(Wn)$,

χ_d (T W$_n$) = $\begin{matrix} 7\ if\ n\ odd \\ 6\ if\ n\ even \end{matrix}$.

Proof

Let $V(Wn) = \{\, v_i, u_i, w_i\ for\ 1 \leq i \leq n\}$ and Let $((Wn)) = \{\, v_i, u_i, w_i\ for\ 1 \leq i \leq n\,\}\ U\ \{\, p_i, a_i, b_i, c_i\ for\ 1 \leq i \leq n\,\}$, where p_i, a_i, b_i, c_i are the vertices of $M(W_n)$ corresponding to the edge $v_i v_{i+1}$ of $W_n(1 \leq i \leq n-1), v_i u_i$ of $W_n(1 \leq i \leq n)$, $u_i u_{i+1}$ of $W_n\ (1 \leq i \leq n-1)$, $u_i w_i$ of $W_n\,(1 \leq i \leq n)$.

Note that any three consecutive vertices of the path must be colored differently in any dynamic coloring of $M(W_n)$, since the first and third vertices are the only neighbors of second vertex and must be colored differently (by double adjacency conditions) and also differently from the second vertex.

Case 1 if n is odd

• For $1 \leq i \leq n$, $\forall\, n$ is odd, assign the color c_1 to vertices v_i

• For $1 \leq i \leq n$, $\forall\, n$ is even, assign the color 2 to vertices v_i and assign the color C$_3$ to vertex v_n

• For $1 \leq i \leq n$, $\forall\, n$ is odd, assign the color c_5 to vertices u_i

• For $1 \leq i \leq n$, $\forall\, n$ is even, assign the color c_6 to vertices u_i and assign the color c_7 to vertex u_n

• For $1 \leq i \leq n$, $\forall\, n$ is odd, assign the color c_6 to vertices w_i

• For $1 \leq i \leq n$, $\forall\, n$ is even, assign the color c_5 to vertices w_i

• For $1 \leq i \leq n$, $\forall\, n$ is odd, assign the color c_3 to vertices p_i

• For $1 \leq i \leq n$, $\forall\, n$ is even, assign the color c_4 to vertices p_i and assign the color c_2 to vertex p_n

• Assign the color c_4 to vertex a_1 and c_1 to vertex a_n

• For $1 \leq i \leq n$, $\forall\, n$ is even, assign the color c_1 to vertices a_i

• For $2 \leq i \leq n-1$, $\forall\, n$ is odd, assign the color c_2 to vertices a_i

• For $1 \leq i \leq n$, $\forall\, n$ is odd, assign the color c_3 to vertices b_i

• For $1 \leq i \leq n$, $\forall\, n$ is even, assign the color c_4 to vertices b_i and assign the color c_2 to vertex b_n

• For $1 \leq i \leq n$, $\forall\, n$ is odd, assign the color c_1 to vertices c_i

• For $1 \leq i \leq n$, $\forall\, n$ is even, assign the color c_2 to vertices c_i and assign the color c_3 to vertex c_n

Hence odd graph has 7 colors.

Case 2 if n is even

• For $1 \leq i \leq n$, $\forall\, n$ is odd, assign the color c_1 to vertices v_i

• For $1 \leq i \leq n$, $\forall\, n$ is even, assign the color c_2 to vertices v_i

• For $1 \leq i \leq n$, $\forall\, n$ is odd, assign the color c_5 to vertices u_i

• For $1 \leq i \leq n$, $\forall\, n$ is even, assign the color c_6 to vertices u_i

• For $1 \leq i \leq n$, $\forall\, n$ is odd, assign the color c_6 to vertices w_i

• For $1 \leq i \leq n$, $\forall\, n$ is even, assign the color c_5 to vertices w_i

• For $1 \leq i \leq n$, $\forall\, n$ is odd, assign the color c_3 to vertices p_i

• For $1 \leq i \leq n$, $\forall\, n$ is even, assign the color c_4 to vertices p_i

• For $1 \leq i \leq n$, $\forall\, n$ is even, assign the color c_1 to vertices a_i

• For $1 \leq i \leq n$, $\forall\, n$ is odd, assign the color c_2 to vertices a_i

• For $1 \leq i \leq n$, $\forall$ n is odd, assign the color c_3 to vertices b_i

• For $1 \leq i \leq n$, $\forall$ n is even, assign the color c_4 to vertices b_i

• For $1 \leq i \leq n$, $\forall$ n is odd, assign the color c_1 to vertices c_i

• For $1 \leq i \leq n$, $\forall$ n is even, assign the color c_2 to vertices c_i

Hence even graph has 6-colors.

Hence, $\chi_d(T\,Wn) = \begin{cases} 7 \text{ if } n \text{ odd} \\ 6 \text{ if } n \text{ even} \end{cases}$, $\forall$ $n \geq 3$. An easy check shows that this is minimum dynamic 5-coloring.

Theorem 4.3 If $n \geq 3$ the dynamic chromatic number of the central graph of web graph $C(W_n)$, $\chi_d(C(W_n)) = 3n$.

Proof

Let v_i, u_i, w_i for $(1 \leq i \leq n)$ be the vertices of W_n. Let p_i, a_i, b_i, c_i be the subdivisions of the edges $v_i v_{i+1}$ of W_n $(1 \leq i \leq n-1)$, $v_i u_i$ of W_n $(1 \leq i \leq n)$, $u_i u_{i+1}$ of W_n $(1 \leq i \leq n-1)$, $u_i w_i$ of W_n $(1 \leq i \leq n)$.

Consider $3n$-coloring of $C(W_n)$ as dynamic.

• For $1 \leq i \leq n$, assign the color c_i to vertices v_i

• For $1 \leq i \leq n$, assign the color c_{n+i} to vertices u_i

• For $1 \leq i \leq n$, assign the color c_{2n+i} to vertices w_i

• For $1 \leq i \leq n-2$, assign the color c_{i+2} to vertices p_i, assign the color c_2 to vertex p_n and c_1 to vertices p_{n-1}.

• If a_i is even

For $1 \leq i \leq n$, $\forall$ n is odd, assign the color c_2 to vertices a_i

For $1 \leq i \leq n$, $\forall$ n is even, assign the color c_3 to vertices a_i

• If a_i is odd

For $1 \leq i \leq n$, $\forall$ n is odd, assign the color c_2 to vertices a_i

For $1 \leq i \leq n$, $\forall$ n is even, assign the color c_3 to vertices a_i and assign the color c_4 to the vertex a_n

• For $1 \leq i \leq n$, assign the color c_1 to vertices b_i

• For $1 \leq i \leq n$, assign the color c_1 to vertices c_i

Thus we get, $\chi_d(C(W_n)) \leq 3n$.

Suppose $\chi_d(C(W_n)) = 3n-1$.

Here each p_i and p_{i+1}, a_i and a_{i+1}, b_i and b_{i+1}, c_i and c_{i+1} are non-adjacent vertices.

A $3n-1$ $-$coloring of $C(W_n)$ in which p_i and p_{i+1}, a_i and a_{i+1}, b_i and b_{i+1}, c_i and c_{i+1} receive the corresponding same colors and must satisfy both adjacency and double adjacency conditions unless the double adjacency condition is not satisfied for some vertex v, u, w adjacent to p_i and p_{i+1}, a_i and a_{i+1}, b_i and b_{i+1}, c_i and c_{i+1}. This is contradiction that the dynamic-coloring with $3n-1$ colors is not possible.

Therefore, $\chi_d(C(W_n)) \geq 3n$.

Hence, $\chi_d\,C\,W_n = 3n$, $\forall$ n ≥ 3.

Theorem 4.4 If $n \geq 3$ the dynamic chromatic number of the line graph of web graph $L(W_n)$, $\chi_d(L(W_n)) = 4$.

Proof

By definition of line graph, each edge of W_n is taken to be as vertex is $L(W_n)$ The vertices $p_i\ a_i\, b_i\ c_i$ for $(1 \leq i \leq n)$ induce a clique of order n in $L(W_n)$, That is $V(L(W_n)) = \{p_i : 1 \leq i \leq n\}\ U\ \{a_i : 1 \leq i \leq n\}\ U\ \{b_i : 1 \leq i \leq n\}\ U\ \{c_i : 1 \leq i \leq n\}$.

Thus we have, $\chi_d(L(W_n)) \geq 4$.

Now consider the vertex set $V(L(W_n))$ and color class $c = \{c_1, c_2, c_3, \ldots, c_n\}$. Assign the colors to $L(W_n)$ to obtain dynamic-coloring as follows:

Case 1 if n is even

• For $1 \leq i \leq n$, $\forall$ n is odd, assign the color c_1 to vertices p_i

• For $1 \leq i \leq n$, $\forall$ n is even, assign the color c_2 to vertices p_i

• For $1 \leq i \leq n$, assign the color c_3 to vertices a_i

• For $1 \leq i \leq n$, $\forall$ n is odd, assign the color c_1 to vertices b_i

• For $1 \leq i \leq n$, $\forall$ n is even, assign the color c_2 to vertices b_i

• For $1 \leq i \leq n$, assign the color c_4 to vertices c_i

Case 2 if n is odd

• For $1 \leq i \leq n$, $\forall$ n is odd, assign the color c_1 to vertices p_i

• For $1 \leq i \leq n$, $\forall$ n is even, assign the color c_2 to vertices p_i and assign the color c_3 to vertex p_n

• Assign the color c_2 to vertex a_1

• For $2 \leq i \leq n-1$, assign the color c_3 to vertices a_i and assign the color c_1 to vertex a_n

• For $1 \leq i \leq n$, $\forall n$ is odd, assign the color c_1 to vertices b_i

• For $1 \leq i \leq n$, $\forall n$ is even, assign the color c_2 to vertices b_i and assign the color c_3 to vertex b_n

• For $1 \leq i \leq n$, assign the color c_4 to vertices c_i

Thus, $\chi_d (L(Wn)) \leq 4$

Hence, $\chi_d (L(W_n)) = 4, \forall n \geq 3$. An easy check shows that this is minimum dynamic 4-coloring.

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