ON THE CONTROLLABILITY OF IMPULSIVE FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT

In this paper, we established the some sufficient conditions for controllability of impulsive functional integrodifferential equations with nonlocal conditions by using the measure of noncompactness and Monch fixed point theorem.

Keywords: Controllability, impulsive functional integro-differential equations, Banach space.

1. INTRODUCTION

Impulsive differential equations are a class of important models which describes many evolution process that abruptly change their state at a certain moment, see the monographs of Bainov and Simeonov [2], Lakshmikantham et al.[8] and have been studied extensively by many authors[3,4,10]. On the other hand, the concept of controllability is of great importance in mathematical control theory. Many authors have been studied the control of nonlinear systems with and without impulses; see for instance[5, 6, 7].

The starting point of this paper is the work in papers [5,9]. Especially, authors in [9] investigated the controllability results of mixed-type functional integrodifferential evolution equations with nonlocal conditions

\[
\begin{align*}
    x'(t) &= Ax(t) + f(t,x(t),x(\sigma(t))) + Bu(t), \\
    x(t_i^+) &= x(t_i^-) + I_i x(t_i),  \\
    x(0) &= x_0,
\end{align*}
\]

where \( A \) is a family of linear operators which generates an evolution operator \( U(t,s) : \mathbb{R} \times [0,b] \to \mathbb{R} \), \( b \) is a Banach space, \( L \) is the space of all bounded linear operators in \( X \).

The paper is organized as follows: In section 2, we will recall some basic notations definition, hypothesis and necessary preliminaries. In section 3, we prove the controllability of impulsive integro-differential system with nonlocal system (1.7) – (1.9), using Monch fixed point theorem.

2. PRELIMINARIES

In this section, we recall some basic definitions and lemmas which will be used to prove our main results of this paper.

Let \((X,\cdot,\cdot)\) be a real Banach space. We denote by \(C([0,b];X)\) the space of \(X\)-valued continuous function on \([0,b]\) with the norm \(\|x\| = \sup_{t \in [0,b]} \|x(t)\|\) and by \(L^p([0,b];X)\) the space of \(X\)-valued Bochner integrable functions on \([0,b]\) with the norm \(\|f\|_{L^p} = \left( \int_0^b \|f(t)\|^p dt \right)^{\frac{1}{p}}\).

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For the sake of simplicity, we put $J = [0, b]$; $J_t = (t_i, t_{i+1})$, $i = 1, \ldots, n$. In order to define the mild solution of problem (1.7)-(1.9), we introduce the set $\text{PC}([0,b];X) = \{ u : [0,b] \rightarrow X : u \text{ is } J \}$; continuous on each bounded subset $\Omega$ of $X$, and the right limit $u(t_i^+)$ exists, $i = 1, \ldots, n$. It is easy to verify that $\text{PC}([0,b];X)$ is a Banach space with the norm $\| u \| = \sup_{t \in [0,b]} | u(t) |$.

**Definition 2.1:** Let $E^+$ be the positive cone of an ordered Banach space $(E, \leq)$. A function $\Phi$ defined on the set of all bounded subsets of the Banach space $X$ with values in $E^+$ is called a measure of noncompactness (MNC) on $X$ if $\Phi(c) = \Phi(\emptyset)$ for all bounded subsets $\Omega \subset X$, where $c$ stands for the closed convex hull of $\Omega$.

The MNC $\Phi$ is said to be:

1. Monotone if for all bounded subsets $\Omega_1, \Omega_2$ of $X$ we have:
   $$\Omega_1 \subseteq \Omega_2 \Rightarrow (\Phi(\Omega_1) \leq \Phi(\Omega_2));$$

2. Nonsingular if $\Phi(\{a\} \cup \Omega) = \Phi(\Omega)$ for every $a \in X$ such that $\{a\} \subseteq X$;

3. Regular if $\Phi(\emptyset) = 0$ if only if $\emptyset$ is relatively compact in $x$.

One of the most important examples of MNC is the noncompactness measure of Hausdorff $\beta$ defined on each bounded subset $\Omega$ of $X$ by $\beta(\Omega) = \inf \{ \varepsilon : \Omega \text{ can be covered by a finite number of balls of radii smaller than } \varepsilon \}$. For all bounded subsets $\Omega_1, \Omega_2$ of $X$,

1. $\beta(\Omega_1 \cup \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$, where $\Omega_1 \cup \Omega_2 = \{x + y : x \in \Omega_1, y \in \Omega_2\}$;

2. $\beta(\Omega_1 \cup \Omega_2) \leq \max\{\beta(\Omega_1), \beta(\Omega_2)\}$;

3. $\beta(\lambda \Omega) = \lambda \beta(\Omega)$ for any $\lambda \in \mathbb{R}$;

4. If the map $Q : D(Q) \subseteq X \rightarrow Z$ is Lipschitz continuous with constants $k, \eta$, then $\beta(\Omega) \leq k\beta(\Omega)$ for any bounded subset $\Omega \subset D(Q)$, where $Z$ is a Banach space.

**Definition 2.2:** A two parameter family of bounded linear operators $U(t, s), 0 \leq s \leq t \leq b$ on $X$ is called an evolution system if the following two conditions are satisfied:

(i) $U(t, s) = I, U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq b$;

(ii) $U(t, s)$ is strongly continuous for $0 \leq s \leq t \leq b$

and there exists $M > 0$ such that $U(t, s) \leq M$ for any $(t, s) \in T$.

**Definition 2.3:** A function $x(\cdot) \in \text{PC}([0,b];X)$ is a mild solution of (1.7)-(1.9) if

$$x(t) = U(t, 0)x_0 + \int_0^t U(t, s)f(s, x_s)\, ds + \int_0^t B(t)u_0\, ds$$

for all $t \in [0, b]$, where $x_0 + M x = x_0$.

**Definition 2.4:** The system (1.7) - (1.9) is said to be controllable on the interval $J$ if for every initial function $\varphi \in D$ and $x_0 \in X$, there exists a control $u \in L_2(J, [0, b])$ such that the mild solution $x(\cdot)$ of (1.7) - (1.9) satisfies $x(\cdot)$.

**Definition 2.5:** A countable set $\{f_n\}_{n=1}^{+\infty} \subset L_1([0,b];X)$ is said to be semicom pact if:

1. The sequence $\{f_n\}_{n=1}^{+\infty}$ is relatively compact in $X$ for $a.e. \ t \in [0, b]$.

2. There is a function $\mu \in L_1([0,b];R^+)$ satisfying $\sup_{n \geq 1} f_n(t) \leq \mu(t)$ for $a.e.$

**Lemma 2.1:** Let $\{f_n\}_{n=1}^{+\infty}$ be a sequence of function in $L_1([0,b];R^+)$. Assume that there exist $\beta : \text{me}L_1([0,b];R^+) \supseteq \{ \mu \geq 0 \} \rightarrow \mathbb{R}$ such that $\beta(\mu) = \sup_{n \geq 1} f_n(t) \leq \mu(t)$ and

$$\beta(\sup_{n \geq 1} f_n(t)) \leq \beta(\mu(t))$$

for $\mu(t) \leq 1$ and $t \in [0, b]$. Then for all $t \in [0, b]$, we have $\beta(\sup_{n \geq 1} f_n(t)) \leq \beta(\mu(t))$.

**Lemma 2.2:** Let $(Gf)(t) = \int_0^t U(t, s)f(s)\, ds$. If $\{f_n\}_{n=1}^{+\infty} \subset L_1([0,b];X)$ is semicom pact then the set $\{Gf_n\}_{n=1}^{+\infty}$ is relatively compact in $C([0,b];X)$ and moreover, if $f_n \rightarrow f_0$ then for all $t \in [0, b]$,

$$(Gf_n) t \rightarrow (Gf_0) t$$

as $n \rightarrow +\infty$.

**Lemma 2.3:** Let $D$ be a closed convex subset of a Banach space $X$ and $0 \in D$. Assume that $F : D \rightarrow X$ is a continuous map which satisfies Monch’s condition, that is, $M \subset \text{Discountable}, M \subseteq \{0\}UF(M) = M$ is compact. Then, there exists $x \in D$ with $x = F(x)$.

### 3. CONTROLLABILITY RESULTS

We first give the following hypothesis:

**H1** $A(t)$ is a family of linear operators $A(t) : D(A) \rightarrow X$, $D(A)$ not depending on $t$ and dense subset of $X$, generating an equicontinuous evolution system $\{U(t,s) : (t,s) \in \Delta\}$, i.e.,

$$(t,s) \rightarrow \{U(t, s) x : x \in B\}$$

is equicontinuous for $t > 0$ and for all bounded subsets $B$.

**H2** The function $f : [0,b] \times X \rightarrow X$ satisfies:

1. For $a.e. \ t \in [0, b]$, the function $f(t,\cdot) : X \rightarrow X$ is continuous and for all $x \in X$, the function $f(\cdot, x) : [0,b] \rightarrow X$ is measurable;

2. If there exists a function $\mu \in L_1([0,b];R^+)$ and a nondecreasing continuous function $\Omega : R^+ \rightarrow R^+$ such that $f(t,x) \leq \mu(t) \Omega(x)$, $x \in X$, $t \in [0,b]$, and

$$\lim_{t \rightarrow +\infty} \Omega(t)$$

then $f(t,x) \leq \mu(t)$ for $t \in [0, b]$. This is called the monotonicity condition.
\[ \inf_{\Omega} \hat{a}_n = 0. \]
(iii) There exists heL^1([0,b];R^+) such that, for any bounded subset D \subset X,
\[ \beta f t, x t \leq t \beta(x(t)) \text{ for a.e. } \]
t \in [0, b], where \beta is the Hausdorff \text{MNC}

(H3) The function h: [0,b] \times X \rightarrow X satisfies:
(i) For each t, se [0,b], the function h(t,s)\rightarrow X is continuous and for all \( x \in X \), the
function h(\cdot; x): [0,b] \rightarrow X is measurable;
(ii) There exists a function \( \mu \in L^1(0,b;\mathbb{R}^+) \) such that
\[ (t,s,x(s)) \leq m t, s x(s), \]
\( x \in X \), \( t, s \in [0,b] \) and \( \lim_{n \to \infty} \inf \frac{\| x \|}{n} = 0. \)

(H4) M: PC(J,X) \rightarrow X is a continuous compact operator such that
\[ \lim_{y \to \infty} \frac{M(y)}{y} = 0; \]
\( t \in J \), \( x \in X \), and for \( i \),
\[ \beta W^1 \tau t \leq K_i t \beta(Q). \]

(H5) The linear operator W:L^2 J, V \rightarrow X is defined by \( Wt = b U b, b \) is measurable and for \( i \)
\[ 0 \leq k \leq \infty \]
\( (i) \) W has an invertible operators \( W^{-1} \), which take values in \( L^2 J, V \ker W \) and there
exist positive constants \( M_2, M_3 \) such that \( B \leq M_2 \) and \( W^{-1} \leq M_3; \)
(ii) there is \( K \in L^1(0,b;\mathbb{R}^+) \) such that \( \beta W^1 \tau t \leq K_i t \beta(Q) \)
\( (H6) \) Let \( I: X \rightarrow X, i = 1, ..., s \) be a continuous operator such that:
(i) There are nondecreasing functions \( I: R^+ \rightarrow R^+, i = 1, ..., s \) such that
\[ l_i(x) \leq l_i x \]
\( i \in [1, s] \) and \( \lim_{n \to \infty} \inf \frac{l_i(x)}{l_i(x)} = 0, i = 1, ..., s. \)
(ii) There exist constants \( K \geq 0 \) such that \( \beta I x t \leq K_i t \beta(x(t)) \). \( i = 1, ..., s. \)

(H7) The following estimation holds true:
\[ \text{L} = \left( M_1 + 2M_2^2K_i \right) i \leq K_i + 4M_1 + \frac{8M_12M_2KW_i}{L_1+K_i} \]
\( b < 1 \)
Where \( M_1 = \sup \{ U t, s, (t,s) \in \Delta \} \)

Theorem: Assume that (H1) – (H7) are satisfied, then
the impulsive integrodifferential system
\[ (1.7)-(1.9) \]
is nonlocally controllable on J, provided
\[ \frac{1}{n} \left[ C_1 + C_2 M(x) + C_3 \Omega + C_4 x(t) + \right. \]
\[ C_5 \left. \frac{d}{d \xi} b(n) \right] \leq 1. \]
Proof: Using hypothesis (H5) (i), for every \( x \in PC(J,X) \), define the control
\[ u(t) = W^{-1} x_t - M x_n - U t, b = 0, x_0 = M x_n \]
\[ \frac{b}{-U b, s \in [0,b], x_n} + s, \tau x(t) d \]
\[ \tau d s \]
\[ \frac{b}{-U t, t, I, x_n, t,} \]
\( \tau \leq \infty \)
We shall show that, when using this control, the operator, defined by
\[ G x t = U(t, 0)(x_0 - M(x)) \]
\[ + U(t, s) f s, x t \]
\[ + (s, \tau x(t)) d \tau + B u(s) d \tau \]
\( \tau \leq \infty \)
has a fixed point. This fixed point is then a solution of the system (1.7)-(1.9). Clearly
\( x b = x_t - M x = G x (b) \) which implies that the system (1.7)-(1.9) is controllable.
We define \( G = G_1 + G_2 \) where
\[ G x t = U(t, 0)(x_0 - M(x)) + \delta_{c \leq t} U t, t, I, x_t(t) \]
\[ \delta_{c \leq t} U t, s \]
\[ + (s, \tau x(t)) d \tau + B u(s) d \tau \]
\( \tau \leq \infty \)
for all \( x, b. \) subsequently, we will prove that \( G \) has a fixed point by using lemma2.3. (Monch fixed point theorem).

Step 1: There exist a positive integer \( n_0 \geq 1 \) such that \( G(B_{n_0}) \subseteq B_{n_0} \), where \( B_{n_0} = \{ x \in PC(J,X), x \leq n_0 \}. \)
Suppose the contrary. Then we can find \( x_0 \in PC(J,X), y_n = G x_n, \epsilon PC(J,X) \) such that \( x_n \leq n \)
and \( y_n \leq n > n \) for every \( n \geq 1 \).
Now we have
\[ y_n(t) = U(t, 0) \left( x_0 - M(x_n) \right) t + U(t, s) f_s x_n s \]

\[ \left( s, \tau, x_n(\tau) \right) d\tau + Bu_{x_n} \]

\[ \sum_{i=1}^{n} \frac{1}{x_n} \]

\[ u_n \leq M \sum_{i=1}^{n} \frac{1}{x_n} \]

\[ u_{x_n}^2 \leq M \frac{1}{x_n} \]

\[ + M_1 I_1 x_n \]

(3.3)

Substituting (3.3) in (3.2) we get

\[ 1 \]

\[ \frac{1}{n} \frac{C_1 + C M x + C \Omega n + C x \tau}{n} \]

\[ + C_5 \]

\[ = \]

\[ w \] where

\[ C_1 = \]

\[ C_2 = \]

\[ C_3 = \]

\[ C_4 = \]

\[ C_5 = \]

\[ M_1 M_2 b M_3 \]

by passing to the limit as \( n \to +\infty \) in (3.4), we get \( I \leq 0 \), which is a contradiction. Thus we deduce that there is \( n_0 \geq 1 \) such that \( G(x_{n_0}) \subseteq B_{n_0} \).

Step 2: The operators \( G \) is continuous on \( PC \ 0, b ; X \) for this purpose, we assume that \( x_0 \to x \) in \( PC \ 0, b ; X \). Then by hypothesis (H4) and (H6), we have

\[ G(x_0) \to G(x) \leq M_1 M x_n - M x \]

\[ + M_1 I_1 x_n t_i \]

\[ - I_1 x t_i \] (3.5)

\[ \geq M_1 f_s x_n s - f_s x s \]

\[ + M_1 b \int_{0}^{s} \left[ \right] d\tau + Bu_{x_n} \]

\[ \left( s, \tau, x(\tau) \right) \]

\[ ds \]

(3.6)

By domination convergence theorem, we have

\[ G x_n \to G x \leq G x_n \to G x \]

\[ + M_1 M x_n - M x \]

\[ + M_1 f_s x_n s - f_s x s \]

\[ + M_1 b \int_{0}^{s} \left[ \right] d\tau + Bu_{x_n} \]

\[ \left( s, \tau, x(\tau) \right) \]

(3.7)

By the equicontinuity property of \( U \), \( s \) and the absolute continuity of the lebesgue integral, the right hand side of the inequality equation (3.8) tends to zero independent of \( y \) as \( t_2 \to t_1 \).

Therefore \( G(D) \) is equicontinuous on every \( J_i \).

Step 4: Assume that \( D = \{ x_n \}_{n=1}^{+\infty} \) since \( G \) maps \( D \) into an equicontinuous family, \( G(D) \) is equicontinuous on \( J_i \). Hence \( D \subseteq co \ 0 \cup G D \) is also equicontinuous on every \( J_i \).

Now we shall show that \( (GD)(t) \) is relatively compact in \( X \) for each \( t \in J \).

From the compactness of \( M(\cdot) \), we have

\[ \beta(G_{x_n}(t)) \leq M_1 K \beta x t_i \]

(3.9)
for \( t \leq 0, b \) by lemma (2.1), we have

\[
\beta_n(s) = \sum_{n=1}^b \beta(\xi(s)) + M_1 K \beta(x(t))
\]

Then this implies that

\[
\beta \left( \sum_{n=1}^b \beta(\xi(s)) + M_1 K \beta(x(t)) \right) \leq \sum_{n=1}^b \beta(\xi(s)) + M_1 K \beta(x(t))
\]

Therefore

\[
\beta \left( \sum_{n=1}^b \beta(\xi(s)) + M_1 K \beta(x(t)) \right) \leq \sum_{n=1}^b \beta(\xi(s)) + M_1 K \beta(x(t))
\]

Finally, due to lemma 2.1, we have

\[
\beta \left( \sum_{n=1}^b \beta(\xi(s)) + M_1 K \beta(x(t)) \right) \leq \sum_{n=1}^b \beta(\xi(s)) + M_1 K \beta(x(t))
\]

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