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RESEARCH ARTICLE

ON *r* - DYNAMIC COLORING OF SOME GRAPHS

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ABSTRACT

The r-dynamic coloring of a graph H is a proper p-coloring of the vertices of the graph H so that for every vertex $a \in V(H)$ has neighbors in atleast $min\{r, d(a)\}$ distinct classes of color. The least p which provides H an r-dynamic coloring with p colors is known as r-dynamic chromatic number of the graph H and it is denoted as $\chi_r(H)$. In this paper, we have attained the lower, upper bound and exact r-dynamic chromatic number for cocktail party graph Cp_s, s-barbell graph Ba_s, windmill graph W_s^q, book graph B_s and pencil graph Pc_s.

Mathematics subject classification: 05C15

Keywords: r-dynamic coloring; cocktail party graph Cp_s; s-barbell graph Ba_s; windmill graph W_s^q; book graph B_s; pencil graph Pc_s.

1. INTRODUCTION

Throughout we take into account simple, finite and connected graphs. Montgomery was the pioneer in dynamic coloring. Dynamic coloring [1, 2, 3, 5, 9, 10] of a graph is proper coloring of H so that each and every vertex $a \in V(H)$ has neighbors in atleast two different classes of color. And its generalized version is r-dynamic coloring. A mapping $c : V(H) \to Q$, the set of colors with $Q = |p|$, is known as r-dynamic coloring if the following two rules holds:

1) $c(a) \neq c(z)\ for\ az \in E(G)$ and
2) $\left|c\big(N(z)\big)\right| \geq min\{r, d(z)\}$, for each and every $z \in V(H)$ where $N(z)$ denotes the set of neighbors of z, r is a positive integer and $d(z)$ is the degree of the vertex z in H.

The first rule is an indication for proper coloring and the second rule is the r-adjacency condition. The r-dynamic chromatic number is the least p that allows H an r-dynamic coloring with p colors and it is denoted as $\chi_r(H)$. The r-dynamic chromatic number does not differ once r reaches the saturation value Δ. There are many open problems one among them was conjectured by Montgomery which states for regular graphs the result $\chi_r(H) \leq \chi_r(H) + 2$. Graph coloring is one among the most challenging problems in mathematics and has many real-life applications.

2. PRELIMINARIES

[7] The **Cocktail Party Graph** Cp_s is a graph with $s = 2q$ vertices $a_j, j = 1, 2, \ldots, 2q$ with a_j non-adjacent to a_{j+q} and adjacent to all other vertices.

[4] The s- **Barbell Graph** Ba_s is attained by connecting two copies of complete graph K_s by a bridge. Here, we are connecting the first two vertices of K_s by a bridge.

[12] The **Windmill Graph** W_s^q with $q, s \geq 2$ is constructed by considering q copies of the complete graph K_s with a universal vertex (common vertex). When $q = 2$ and $q = 3$ i.e., W_s^2 and W_s^3 they are the star graph and friendship graph respectively.

[8] The **Book Graph** B_s is the Cartesian product of star graph $K_{1,s}$ and path P_2 i.e., $K_{1,s} \times P_2$.

[6] For $s \geq 2$, the **Pencil Graph** Pc_s is a graph with $2s + 2$ vertices where the vertex set is $\{a_q, b_q : q = 0,1, \ldots, s\}$ and edge set $\{a_q a_{q+1}, b_q b_{q+1} : q = 1, 2, \ldots, s - 1\} \cup \{a_0 a_1, a_0 b_1, b_0 a_s, b_0 b_s\} \cup \{a_q b_q : q = 0, 1, \ldots, s\}$.

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3. OBSERVATIONS

Lemma 3.1. $\chi_r(H) \geq min\{r, \Delta(H)\} + 1$ is a lemma providing the lower bound for *r*-dynamic chromatic number found by Montgomery and Lai [9].

Note 3.2. We can observe easily from the graph Cp_s *that there is a clique of order* $\frac{s}{2} = q$ *hence* $\chi_r(Cp_s) \geq q$.

Note 3.3. From the definition of Ba_s and W_s^q there is a maximal complete subgraph of order *s* hence $\chi_r(Ba_s) \geq s$ and $\chi_r\left(W_s^q\right) \geq s$.

4. RESULTS

Lemma 4.1. For $q \geq 2$, the lower bound for *r*-dynamic chromatic number of cocktail party graph Cp_s *is,* $\chi_r(Cp_s) \geq \begin{cases} q, 1 \leq r \leq q-1 \\ r+1, q \leq r \leq \Delta(Cp_s) \end{cases}$

Proof. The cocktail party graph is Cp_s is regular graph with degree 2(q - 1). Then $V(Cp_s) = \{a_j, j = 1, 2, \dots, 2q\}$ and $E(Cp_s) = \{a_j a_k: j, k = 1, 2, \dots, 2q \text{ where } j \neq k = j + q\}$. *Also,* $\delta(Cp_s) = \Delta(Cp_s) = 2(q-1)$. The order of Cp_s *is* $|V(Cp_s)| = s = 2q$ and size is $|E(Cp_s)| = s(q-1)$.

By Note 3.2 we have $\chi_{1 \leq r \leq q-1}(Cp_s) \geq q$.

For $q \leq r \leq \Delta(Cp_s)$, by Lemma 3.1 $\chi_r(H) \geq min\{r, \Delta(H)\} + 1$.

Therefore,

$$\chi_{q \leq r \leq \Delta(Cp_s)}(Cp_s) \geq min\{r, \Delta(Cp_s)\} + 1 = r + 1.$$

Theorem 4.2. For $q \geq 2$, the *r*-dynamic chromatic number of cocktail party graph Cp_s *is,* $\chi_r(Cp_s) = \begin{cases} q, 1 \leq r \leq q-1 \\ 2q, q \leq r \leq \Delta(Cp_s) \end{cases}$

Proof. We have two cases: $1 \leq r \leq q-1$ *and* $q \leq r \leq \Delta(Cp_s)$ to consider.

Case 1: When $1 \leq r \leq q-1$.

By Lemma 4.1 we have the lower bound $\chi_{1 \leq r \leq q-1}(Cp_s) \geq q$. Consider the map $c_1: V(Cp_s) \rightarrow \{1,2,\dots,q\}$ and coloring as follows:

$c_1(a_j) = j, j = 1,2, \dots, q$.

$c_1(a_j) = k, j = q + k \text{ and } k = 1,2,\dots,q$ since a_j and a_{j+q} are non-adjacent.

By the above coloring $\chi_{1 \leq r \leq q-1}(Cp_s) \leq q$. Hence $\chi_{1 \leq r \leq q-1}(Cp_s) = q$.

Case 2: When $q \leq r \leq \Delta(Cp_s)$.

By Lemma 4.1 we have $\chi_{q \leq r \leq \Delta(Cp_s)}(Cp_s) \geq r + 1$ but inorder to satisfy the *r*-adjacency condition we require 2q colors in total hence we the lower bound have $\chi_{q \leq r \leq \Delta(Cp_s)}(Cp_s) \geq 2q$. The upper bound is given by the coloring below considering the mapping $c_2: V(Cp_s) \rightarrow \{1,2,\dots,2q\}$.

$c_1(a_j) = j, j = 1,2, \dots, 2q \text{ and } \chi_{q \leq r \leq \Delta(Cp_s)}(Cp_s) \leq 2q$.

Therefore, $\chi_{q \leq r \leq \Delta(Cp_s)}(Cp_s) = 2q$.

Lemma 4.3. For $s \geq 2$, the lower bound for *r*-dynamic chromatic number of barbell graph Ba_s *is,* $\chi_r(Ba_s) \geq \begin{cases} s, 1 \leq r \leq s-1 \\ s+1, r \geq \Delta(Ba_s) \end{cases}$

Proof. The vertex set of barbell graph $V(Ba_s) = \{a_j, b_j: 1 \leq j \leq s\}$. Here we assume that the vertices a_1 and b_1 are adjacent by a bridge and $E(Ba_s) = \{a_j a_k, b_j b_k : \text{j, k} = 1, 2, \dots, s \text{ and j} \neq \text{k}\} \cup \{a_1 b_1\}$. Also, $\delta(Ba_s) = s - 1$ and $\Delta(Ba_s) = d(a_1) = d(b_1) = s$. And, order of Ba_s *is* $|V(Ba_s)| = 2s$ and size is $|E(Ba_s)| = s^2 - s + 1$.

By Note 3.3 we have $\chi_{1 \leq r \leq s-1}(Ba_s) \geq s$.

For $r \geq \Delta(Ba_s)$, by Lemma 3.1 $\chi_r(H) \geq min\{r, \Delta(H)\} + 1$.

Therefore,

$$\chi_{r \geq \Delta(Ba_s)}(Ba_s) \geq min\{r, \Delta(Ba_s)\} + 1 = \Delta(Ba_s) + 1 = s + 1.$$

Theorem 4.4. For $s \geq 2$, the *r*-dynamic chromatic number of barbell graph Ba_s *is,* $\chi_r(Ba_s) = \begin{cases} s, 1 \leq r \leq s-1 \\ s+1, r = \Delta(Ba_s) \end{cases}$

Proof. We have two cases: $1 \leq r \leq s-1$ and $r = \Delta(Ba_s)$ to consider.

Case 1: When $1 \leq r \leq s-1$.

By Lemma 4.3 we have the lower bound $\chi_{1 \leq r \leq s-1}(Ba_s) \geq s$. The coloring is provided by the mapping $c_3: V(Ba_s) \rightarrow \{1,2,\dots,s\}$ as follows:

$c_3(a_1, a_2, \dots, a_s) = \{1, 2, \dots, s\}$.

$c_3(b_1, b_2, \dots, b_s) = \{2, 3, \dots, s, 1\}$.

By the above coloring $\chi_{1 \leq r \leq s-1}(Ba_s) \leq s$.

Hence $\chi_{1 \leq r \leq s-1}(Ba_s) = s$.

Case 2: When $r = \Delta(Ba_s)$.

By Lemma 4.3 we have the lower bound $\chi_{r=\Delta(Ba_s)}(Ba_s) \geq s+1$. The upper bound is given by the coloring below considering the mapping $c_4: V(Ba_s) \to \{1,\ldots,s+1\}$.

$c_4(a_1, a_2, \ldots, a_s) = \{1, 2, \ldots, s\}$

$c_4(b_1) = s+1$ and $c_3(b_2, b_3, \ldots, b_s) = \{2, 3, \ldots, s\}$.

$\chi_{r=\Delta(Ba_s)}(Ba_s) \leq s+1$.

Therefore, $\chi_{r=\Delta(Ba_s)}(Ba_s) = s+1$.

Lemma 4.5. For $q, s \geq 2$, the lower bound for *r*-dynamic chromatic number of windmill graph W_s^q *is,* $\chi_r(W_s^q) \geq \begin{cases} s, 1 \leq r \leq s-1 \\ r+1, s \leq r \leq \Delta(W_s^q) \end{cases}$

Proof. The vertex set of windmill graph $V(W_s^q) = \{a_0\} \cup \{a_{j,1}, a_{j,2}, \ldots, a_{j,s-1} : 1 \leq j \leq q\}$ where a_0 is universal vertex adjacent to all other vertices $\{a_{j,k} : 1 \leq j \leq q \; and \; 1 \leq k \leq s-1\}$. Edge set is $E(W_s^q) = \{a_{j,k}a_{j,i} : \mathrm{k} \neq \mathrm{i} \;\&\; 1 \leq j \leq q \; and \; 1 \leq k, i \leq s-1\} \cup \{a_0 a_{j,k} : 1 \leq j \leq q \; and \; 1 \leq k \leq s-1\}$. Also, $\delta(W_s^q) = \mathrm{s}-1$ and $\Delta(W_s^q) = \mathrm{d}(a_0) = \mathrm{q}(\mathrm{s}-1)$. And, order of W_s^q *is* $|V(W_s^q)| = \mathrm{q}(\mathrm{s}-1)+1$ and size is $|E(W_s^q)| = \frac{\mathrm{qs}(\mathrm{s}-1)}{2}$.

By Note 3.3 we have $\chi_{1 \leq r \leq s-1}(W_s^q) \geq s$.

For $s \leq r \leq \Delta(W_s^q)$, by Lemma 3.1 $\chi_r(H) \geq min\{r, \Delta(H)\}+1$.

Therefore,

$$\chi_{s \leq r \leq \Delta(W_s^q)}(W_s^q) \geq min\{r, \Delta(W_s^q)\}+1 = r+1.$$

Theorem 4.6. For $q, s \geq 2$, the *r*-dynamic chromatic number of windmill graph W_s^q *is,* $\chi_r(W_s^q) = \begin{cases} s, 1 \leq r \leq s-1 \\ r+1, s \leq r \leq \Delta(W_s^q) \end{cases}$

Proof. We have two cases: $1 \leq r \leq s-1$ and $s \leq r \leq \Delta(W_s^q)$ to consider.

Case 1: When $1 \leq r \leq s-1$.

By Lemma 4.5 we have the lower bound $\chi_{1 \leq r \leq s-1}(W_s^q) \geq s$. The coloring is provided by the map $c_5: V(W_s^q) \to \{1,2,\ldots,s\}$ as follows:

$c_5(a_0) = 1$.

$c_5(a_{j,1}, a_{j,2}, \ldots, a_{j,s-1}) = \{2, 3, \ldots, s\}$ for $1 \leq j \leq q$.

By the above coloring $\chi_{1 \leq r \leq s-1}(W_s^q) \leq s$. Hence $\chi_{1 \leq r \leq s-1}(W_s^q) = s$.

Case 2: When $s \leq r \leq \Delta(W_s^q)$.

By Lemma 4.5 we have $\chi_{s \leq r \leq \Delta(W_s^q)}(W_s^q) \geq r+1$. The upper bound is given by the coloring below considering the mapping $c : V(W_s^q) \to \{1,2,\ldots,r+1\}$ for different stages of *r*.

$c\,(a_0) = 1$ for all cases of *r.*

When *r* = s.

$c\,(a_{1,1}, a_{1,2}, \ldots, a_{1,s-1}) = \{2, 3, \ldots, s\}$

$c\,(a_{2,1}, a_{2,2}, \ldots, a_{2,s-1}) = \{s+1, 3, 4, \ldots, s\}$

$c\,(a_{j,1}, a_{j,2}, \ldots, a_{j,s-1}) = \{2, 3, \ldots, s\} \; for \; 3 \leq j \leq q$

Hence $\chi_{r=s}(W_s^q) \leq s+1$.

When *r* = s+1.

$c\,(a_{2,1}, a_{2,2}, \ldots, a_{2,s-1}) = \{s+1, s+2, 4, \ldots, s\}$

$c\,(a_{j,1}, a_{j,2}, \ldots, a_{j,s-1}) = \{2, 3, \ldots, s\} \; for \; 1 \leq j \leq q \;\; and \;\; j \neq 2.$

Hence $\chi_{r=s+1}(W_s^q) \leq s+2$.

Proceeding like this at each stage of *r* we introduce the color *r*+1 to the next vertex in the list till $a_{j,s-1}$.

Hence $\chi_{s \leq r \leq \Delta(W_s^q)}(W_s^q) \leq r+1$.

Therefore, $\chi_{s \leq r \leq \Delta(W_s^q)}(W_s^q) = r+1$.

Lemma 4.7. For $s \geq 2$, the lower bound for *r*-dynamic chromatic number of book graph B_s is, $\chi_r(B_s) \geq \begin{cases} 2, r = 1 \\ r+1, 2 \leq r \leq \Delta(B_s) \end{cases}$

Proof. The vertex set of book graph $V(B_s) = \{b_{1,j}, b_{2,j}, x, y : 1 \leq j \leq s\}$ and edge set $E(B_s) = \{xb_{1,j}, yb_{2,j}, b_{1,j}b_{2,j} : 1 \leq j \leq s\}$. Also, $\delta(B_s) = 2$ and $\Delta(B_s) = \mathrm{d}(x) = \mathrm{d}(y) = \mathrm{s}$. And, order of B_s *is* $|V(B_s)| = 2\mathrm{s}+2$ and size is $|E(B_s)| = 3\mathrm{s}$.

Since the maximal complete subgraph is of order 2 we have $\chi_{r=1}(Ba_s) \geq 2$.

For $2 \leq r \leq \Delta(B_s)$, by Lemma 3.1 $\chi_r(H) \geq min\{r, \Delta(H)\}+1$.

Therefore, $\chi_{2 \leq r \leq \Delta(B_s)}(B_s) \geq min\{r, \Delta(Ba_s)\}+1 = r+1$.

Theorem 4.8. For $s \geq 2$, the r-dynamic chromatic number of book graph B_s *is,* $\chi_r(B_s) = \begin{cases} 2, \ r = 1 \\ 4, r = 2,3 \\ r + 1, 4 \leq r \leq \Delta(B_s) \end{cases}$

Proof. We have two cases: $r = 1, r = 2, 3$ and $4 \leq r \leq \Delta(B_s)$.

Case 1: When $r = 1$.

By Lemma 4.7 we have the lower bound $\chi_1(Ba_s) \geq 2$. The coloring is provided by the map

$c_6: V(B_s) \to \{1,2\}$ as follows:

$c_6(x) = 1$ and $c_6(y) = 2$.

$c_6(b_{1,j}) = 2$ and $c_6(b_{2,j}) = 1$ for $1 \leq j \leq s$.

By the above coloring $\chi_1(B_s) \leq 2$. Hence $\chi_1(B_s) = 2$.

Case 2: When $r = 2,3$.

By Lemma 4.7 we have the lower bound $\chi_{r=2}(B_s) \geq r + 1 = 3$. But since there is presence of C_4 in B_s which leads to the need of an extra color when $r = 2$. So, the lower bound becomes $\chi_{r=2}(B_s) \geq 4$ and by the same lemma we have the lower bound $\chi_{r=3}(B_s) \geq r + 1 = 4$. Now we assign coloring by the mapping $c_7: V(B_s) \to \{1,2,3,4\}$.

$c_7(x) = 1$ and $c_7(y) = 2$.

$c_7(b_{1,j}) = \begin{cases} 3, j \ is \ odd \\ 4, j \ is \ even \end{cases}$ and $c_7(b_{2,j}) = \begin{cases} 4, j \ is \ odd \\ 3, j \ is \ even \end{cases}$

For $r = 3$ the coloring provided above is sufficient.

So, we have the upper bound $\chi_{r=2,3}(B_s) \leq 4$.

Therefore, $\chi_{r=2,3}(B_s) = 4$.

Case 3: When $4 \leq r \leq \Delta(B_s)$.

By Lemma 4.7 we have the lower bound $\chi_{4 \leq r \leq \Delta(B_s)}(B_s) \geq r + 1$. Consider the mapping

$c : V(B_s) \to \{1,2,\ldots,r+1\}$ which gives the coloring for the vertices.

$c\,(x) = 1$ and $c\,(y) = 2$.

$c\,(b_{1,1}, b_{1,2}, \ldots, b_{1,s}) = \{3, 4, \ldots, r + 1, \underbrace{3, 4, \ldots}_{s-(r-1) \ terms}\}$.

$c\,(b_{2,1}, b_{2,2}, \ldots, b_{2,s}) = \{4, \ldots, r + 1, 3, \underbrace{4, 5, \ldots}_{s-(r-1) \ terms}\}$.

Hence $\chi_{4 \leq r \leq \Delta(B_s)}(B_s) \leq r + 1$. Therefore, we have $\chi_{4 \leq r \leq \Delta(B_s)}(B_s) = r + 1$.

Lemma 4.9. For $s \geq 2$, the lower bound for r-dynamic chromatic number of pencil graph Pc_s *is,*

$\chi_r(Pc_s) \geq \begin{cases} 3, r = 1,2 \\ 4, r \geq \Delta(Pc_s) \end{cases}$

Proof. The pencil graph Pc_s is *a regular graph with degree 3. The vertex set is defined as* $\{a_q, b_q : q = 0,1,\ldots,s\}$ *and* edge set $\{a_q a_{q+1}, b_q b_{q+1} : q = 1, 2, \ldots, s - 1\} \cup \{a_0 a_1, a_0 b_1, b_0 a_s, b_0 b_s\} \cup \{a_q b_q : q = 0, 1, \ldots, s\}$. Also, $\delta(Pc_s) = \Delta(Pc_s) = 3$. And, order of Pc_s *is* $|V(Pc_s)| = 2s + 2$ and size is $|E(Pc_s)| = 3(s + 1)$.

There is a clique of order 3 we have $\chi_{r=1,2}(Pc_s) \geq 3$.

For $r \geq \Delta(Pc_s)$, by Lemma 3.1 $\chi_r(H) \geq min\{r, \Delta(H)\} + 1$.

Therefore, $\chi_{r \geq \Delta(Pc_s)}(B_s) \geq min\{r, \Delta(Pc_s)\} + 1 = \Delta(Pc_s) + 1 = 4$.

Theorem 4.10. For $s \geq 2$, the r-dynamic chromatic number of pencil graph Pc_s *is,* $\chi_r(Pc_s) = \begin{cases} 3, r = 1,2 \ \ s \equiv 0,2(mod\ 3), \\ \ \ r = 1 \ and \ s \equiv 1(mod\ 3) \\ 4, r = 2 \ and \ s \equiv 1(mod\ 3) \\ 5, r = 3 \ and \ \ s \equiv 0(mod 4) \\ 6, \ r = 3 \ and \ otherwise \end{cases}$

Proof. *We have four cases to consider here.*

Case 1: When $r = 1,2 \ and \ s \equiv 0,2(mod\ 3), r = 1 \ and \ s \equiv 1(mod\ 3)$.

Subcase 1: When $r = 1,2 \ and \ s \equiv 0,2(mod\ 3)$

By Lemma 4.9 we have the lower bound $\chi_{r=1,2}(Pc_s) \geq 3$. For upper bound consider the map $c_8: V(Pc_s) \to \{1,2,3\}$.

$c_8(a_0) = 3$

$c_8(a_1, a_2, \ldots, a_s) = \{1, 2, 3, 1, 2, 3, \ \ldots, 1, 2, 3\}$ when $s \equiv 0(mod\ 3)$

$c_8(a_1, a_2, \ldots, a_s) = \{1, 2, 3, 1, 2, 3, \ \ldots, 1, 2\}$ when $s \equiv 2(mod\ 3)$

$c_8(b_1, b_2, \ldots, b_s) = \{3, 2, 1, 3, 2, 1, \ \ldots, 3, 2, 1\}$ when $s \equiv 0(mod\ 3)$

$c_8(b_1, b_2, \ldots, b_s) = \{3, 2, 1, 3, 2, 1, \ \ldots, 3, 2\}$ when $s \equiv 2(mod\ 3)$

$$c_8(b_0) = \begin{cases} 2, when\ s \equiv 0(mod\ 3) \\ 1, when\ s \equiv 2(mod\ 3) \end{cases}$$

Hence $\chi_{r=1,2}(Pc_s) \leq 3$ and therefore $\chi_{r=1,2}(Pc_s) = 3$ when $s \equiv 0,2(mod\ 3)$.

Subcase 2: When $r = 1\ and\ s \equiv 1(mod\ 3)$.

By Lemma 4.9 we have the lower bound $\chi_{r=1}(Pc_s) \geq 3$. Consider the map $c_8: V(Pc_s) \rightarrow \{1,2,3\}$.

$c_8(a_1, a_2, \ldots, a_s) = \{1, 2, 1, 2, \ \ldots, 1, 2\}$ when $s\ is\ even$

$c_8(a_1, a_2, \ldots, a_s) = \{1, 2, 1, 2, \ \ldots, 1\}$ when $s\ is\ odd$

$c_8(b_1, b_2, \ldots, b_s) = \{2, 1, 2, 1, \ \ldots, 2, 3\}$ when $s\ is\ even$

$c_8(b_1, b_2, \ldots, b_s) = \{\ 2, 1, 2, 1, \ \ldots, 2,1,3\}$ when $s\ is\ odd$

$c_8(a_0) = 3$ and $c_8(b_0) = \begin{cases} 1, when\ s\ is\ even \\ 2, when\ s\ is\ odd \end{cases}$

Hence $\chi_{r=1}(Pc_s) \leq 3$ and therefore $\chi_{r=1}(Pc_s) = 3$ when $s \equiv 1(mod\ 3)$.

Case 2: When $r = 2\ and\ s \equiv 1(mod\ 3)$.

By Lemma 4.9 we have the lower bound $\chi_{r=2}(Pc_s) \geq 3$. But we need an extra color when $s \equiv 1(mod\ 3)$ to satisfy *r*-adjacency condition. So, the lower bound transforms to $\chi_{r=2}(Pc_s) \geq 4$.

For upper bound assign the following coloring provided with the mapping $c_9: V(Pc_s) \rightarrow \{1,2,3,4\}$.

$c_9(a_0) = 3$ and $c_9(b_0) = 4$

$c_9(a_1, a_2, \ldots, a_s) = \{1, 2, 3, 1, 2, 3, \ \ldots\}$

$c_9(b_1, b_2, \ldots, b_s) \ = \{2, 3, 1, 2, 3, 1, \ \ldots\}$

Hence $\chi_{r=2}(Pc_s) \leq 4$. Therefore, $\chi_{r=2}(Pc_s) = 4$ when $s \equiv 1(mod\ 3)$.

Case 3: When $r = 3\ and\ s \equiv 0(mod\ 4)$.

By Lemma 4.9 we have the lower bound $\chi_{r=3}(Pc_s) \geq r + 1 = 4$. But inorder to *r*-adjacency condition we need an extra color and hence the lower bound transforms to $\chi_{r=3}(Pc_s) \geq 5$.

Assign the coloring by the map $c_{10}: V(Pc_s) \rightarrow \{1,2, \ldots ,5\}$.

For $1 \leq q \leq s$, $c_{10}(a_q) = \begin{cases} 1, when\ q \equiv 1(mod\ 4) \\ 4, when\ q \equiv 2(mod\ 4) \\ 3, when\ q \equiv 3(mod\ 4) \\ 2, when\ q \equiv 0(mod\ 4) \end{cases}$ $c_{10}(b_q) = \begin{cases} 2, when\ q \equiv 1(mod\ 4) \\ 5, when\ q \equiv 2(mod\ 4) \\ 1, when\ q \equiv 3(mod\ 4) \\ 4, when\ q \equiv 0(mod\ 4) \end{cases}$

$c_{10}(a_0) = 3$ and $c_{10}(b_0) = 5$.

Thus, the upper bound is $\chi_{r=3}(Pc_s) \leq 5$. Therefore, $\chi_{r=3}(Pc_s) = 5$.

Case 4: When $r = 3\ and\ otherwise$.

By Lemma 4.9 we have the lower bound $\chi_{r=3}(Pc_s) \geq 4$. But inorder to *r*-adjacency condition we are forced to introduce two new colors and hence the lower bound $\chi_{r=3}(Pc_s) \geq 6$. Assign the coloring by the map $c_{11}: V(Pc_s) \rightarrow \{1,2, \ldots ,6\}$.

$c_{11}(a_1, a_2, \ldots, a_s) = \{1, 4, 2, 5, 1, 4, 2, 5, \ldots, 1\}$ when $s \equiv 1(mod\ 4)$

$c_{11}(a_1, a_2, \ldots, a_s) = \{1, 4, 2, 5, 1, 4, 2, 5, \ldots, 1, 4\}$ when $s \equiv 2(mod\ 4)$

$c_{11}(a_1, a_2, \ldots, a_s) = \{1, 4, 2, 5, 1, 4, 2, 5, \ldots, 1, 4, 2\}$ when $s \equiv 3(mod\ 4)$

$c_{11}(b_1, b_2, \ldots, b_s) = \{2, 5, 1, 4, 2, 5, 1, 4, \ldots, 2\}$ when $s \equiv 1(mod\ 4)$

$c_{11}(b_1, b_2, \ldots, b_s) = \{2, 5, 1, 4, 2, 5, 1, 4, \ldots, 2, 5\}$ when $s \equiv 2(mod\ 4)$

$c_{11}(b_1, b_2, \ldots, b_s) = \{2, 5, 1, 4, 2, 5, 1, 4, \ldots, 2, 5, 1\}$ when $s \equiv 3(mod\ 4)$

$c_{11}(a_0) = 3$ and $c_{11}(b_0) = 6$.

Thus, the upper bound is $\chi_{r=3}(Pc_s) \leq 6$.

Therefore, $\chi_{r=3}(Pc_s) = 6$, otherwise i.e., when $s \equiv 1,2,3(mod\ 4)$.

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