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EXISTENCE OF SOLUTIONS FOR NEUTRAL FUNCTIONAL VOLTERRA-FREDHOLM INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, we study the existence of mild solutions of nonlinear neutral functional Volterra-Fredholm integrodifferential equations with nonlocal conditions. The results are obtained by using fractional power of operators and Sadovskii's fixed point theorem.

Keywords: Integrodifferential equations, fixed point theorem, Banach space.

MSC subject classification : 45J05, 47H10, 46Bxx.

1. INTRODUCTION

Many phenomena in several branches have mathematical model in terms of differential equations. Differential equations are like a bridge which links mathematics and science with applications. It is a rightly considered as a language of sciences. Many branches of science have led to some kind of differential equations.

The study of integrodifferential equations has emerged in recent years as an independent branch of modern research because of its connections to many applied fields such as elasticity, biology, epidemics and other branches of science and engineering. Neutral differential equations arises in many areas of applied mathematics and for this reasons this equations have received much attention in the last few decades.

The advantages of using nonlocal conditions is that measurable at more places can be incorporated to get better models. The nonlocal Cauchy problem for abstract evolution differential equation was first considered by Byszewski (Byszwski, 1991) Subsequently, several authors have investigated the problem for different types of nonlinear differential equations and integrodifferential equations including functional differential equations in Banach spaces (Balachandran, 1998; Byszwski and Acka, 1998; Balachandran and Park, 2001a, b; Fu and Ezzinbi, 2004).

In the past several years theorems about existence, uniqueness and stability of differential and functional differential abstract evolution Cauchy problem have been studied by Byszewski and Lakshmikantham (1990), Byszewski (1997, 1998), Balachandran and Chandrasekaran (1996), Lin and Liu (1996) and Murugesu and Suguna (2010).

In this paper, we extend this problem to neutral functional Volterra-Fredholm type integro differential equations with nonlocal conditions and discuss the existence of solutions for nonlinear neutral functional Volterra-Fredholm integro differential equations with nonlocal conditions of the form

$$\frac{d}{dt}\left[x(t) + F(t, x(t), x(b_1(t)), \cdots, x(b_m(t)))\right] = Ax(t) + G(t, x(t), x(a_1(t)), \cdots, x(a_n(t))) + K\left(t, x(t), \int_0^t k(t, s, x(s))ds, \int_0^a h(t, s, x(s))ds\right), \quad 0 \le t \le a,$$
$$x(0) = x_0 + g(x) \qquad \text{------------------(1)}$$

where $-A$ generates an analytic semigroup and F, G, K, k, h are given functions to specifed later.

This paper has the following subsections. In section 2, we present some preliminary lemmas and definitions which will be used to prove our main results. In section 3, we present the existence of mild solution of the system (1) using Sadovskii's fixed point theorem (Sodovskii, 1967).

2 PRELIMINARIES

Throughout this work, let $-A$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators T(t) defined in the Banach spaces X. Let $0 \in \rho$ then define the fractional power A^{α}, for $0 \le \alpha \le 1$, as a closed linear operator on its domain $D(A^{\alpha})$ which is dense in X. Further, D(A) is a Banach space under the norm

$$\| x \|_{\alpha} = \| A^{\alpha} x \|, \; x \in D(A^{\alpha})$$

Which we denote by X_{α}. Then for each $0 < \alpha \le 1$, $X_{\alpha} \rightarrow X_{\beta}$ for $0 < \beta < \alpha \le 1$ and the imbedding is compact whenever the resolvent operator of A is compact. We assume that

a) There is a $M \geq 1$ such that $\| T(t) \| \leq M$, for all $0 \leq t \leq a$.

b) For any a > 0, there exists a positive constant C_α,

$$\| A^{\alpha} T(t) \| \leq \frac{C_\alpha}{t^{\alpha}}, \quad 0 < t \leq a \qquad (2)$$

$$\| A^{\beta} F(s_1, x_0, x_1, \cdots, x_m) - A^{\beta} F(s_2, \bar{x}_0, \bar{x}_1, \cdots, \bar{x}_m) \| \leq L\left(| s_1 - s_2 | + \max_{i=0,1,\cdots,m} \| x_i - \bar{x}_i \| \right)$$

for any $0 \leq s_1, s_2 \leq a$, $x_i, \bar{x}_i \in X$, i = 0, 1, ..., m and the inequality

$$\| A^{\beta} F(t, x_0, x_1, \cdots, x_m) \| \leq L_1 (\max\{\| x_i \| : i = 0,1,\cdots, m\} + 1)$$

---------------- (3)

holds for any $(t, x_0, x_1, \ldots, x_m) \in [0,a] \times X^{m+1}$.

(H2) The function $G : [0,a] \times X^{n+1} \to X$ satisfies the following conditions :

(i) For each $t \in [0,a]$, the function $G(t, .) : \times X^{n+1} \to X$ is continuous and for each $(x_0, x_1, \ldots, x_n) \in X^{n+1}$ the function $G(., x_0, x_1, \ldots, x_n) : [0,a] \to X$ is strongly measurable.

(ii) For each positive number $n \in N$, there is a positive function $\phi_n \in L^1([0,a])$ such that

$$\sup_{\|x_0\|,\cdots,\|x_n\| \leq n} \| G(t, x_0, x_1, \cdots, x_n) \| \leq \phi_n(t)$$

$$\lim_{n\to\infty} \frac{1}{n} \int_0^a \phi_n(s) ds = \gamma < \infty$$

(H3) The function $K : [0,a] \times X \times X \times X \to X$ satisfies the following conditions:

(i) For each $t \in [0,a]$, the function $K(t, ., ., .): X \times X \times X \to X$ and for each $x, y, z \in X$, $K(., x, y, z) : [0,a] \to X$ is strongly measurable.

(ii) For each positive number $n \in N$, there exists a positive function $q_n \in L^1([0,a])$ such that

$$\sup_{\|x\| \leq n} \left\| K\left(s, x(s), \int_0^s k(s,\tau, x(\tau)) d\tau, \int_0^a h(s,\tau, x(\tau)) d\tau \right) \right\| \leq q_n(s)$$

$$\text{and} \qquad \lim_{n\to\infty} \frac{1}{n} \int_0^a q_n(s) ds = \gamma_1 < \infty$$

(H4) $a_i, b_j \in C([0,a]; [0,a])$, i = 1, ..., n, j = 1, ..., m. $g \in C(E ; X)$, here after E =

C([0,a]; X) and g is completely continuous.

(H5) There exist positive constants M_3 and M_4 such that

$\| g(x) \| \leq M_3 \|x\| + M_4$ for every $x \in E$.

For our convenience, let us take

$F(0, x(0), x(b_1(0)), \ldots x(b_m(t))) = 0$.

Let $M_0 = \| A^{-\beta} \|$ and also assume the following hypotheses:

(H1) $F : [0,a] \times X^{m+1} \to X$ is a continuous functions and there exists a $\beta \in (0,1)$ and $L, L_1 > 0$ such that the function $A^{\beta}F$ satisfies the Lipschitz condition:

DEFINITION : 2.1 (Pazy, 1983)

Let X be a Banach space, a one parameter family T(t), $0 \leq t < +\infty$, of bounded linear operators from X to X is a semigroup of bounded linear operators on X, if

(i) T(0)=I, where I is the identity operator on X,

(ii) T(t+s)=T(t)T(s) for every $t, s \geq 0$, (the semigroup property)

A semigroup of bounded linear operator T(t) is uniformly continuous if

$$\lim_{t \downarrow 0} \| T(t) - I \| = 0$$

THEOREM : 2.1. (SADOVSKII'S FIXED POINT THEOREM) (Sodovskii, 1967)

Let ψ be a condensing operator on a Banach space X, that is ψ is continuous and takes bounded sets into bounded sets and $\mu(\psi(B)) \leq \mu(B)$ for every bounded set B of X with $\mu(B) > 0$. If $\psi(T) \subset T$ for a convex closed and bounded set γ of X, then ψ has a fixed point in X.

3. EXISTENCE OF MILD SOLUTION

DEFINITION 3.1.

A continuous function $x(.) : [0,a] \to X$ is said to be a mild solution of the Cauchy problem (1), if the function $AT(t-s)F(s, x(s), x(b_1(s)), \ldots, x(b_m(s)))$, $s \in [0,a)$ is integrable on [0,a) and the integral equation is satisfied.

$$\begin{aligned} x(t) = {} & T(t)\left[x_0 + F(0, x(0), x(b_1(0)), \cdots, x(b_m(0))) + g(x)\right] - F(t, x(t), x(b_1(t)), \cdots, x(b_m(t))) \\ & - \int_0^t AT(t-s) F(s, x(s), x(b_1(s)), \cdots, x(b_m(s))) ds \\ & + \int_0^t T(t-s) G(s, x(s), x(a_1(s)), \cdots, x(a_m(s))) ds \\ & + \int_0^t T(t-s) K\left(s, x(s), \int_0^s k(s,\tau, x(\tau)) d\tau, \int_0^a h(s,\tau, x(\tau)) d\tau \right) ds \end{aligned}$$

------------------- (4)

THEOREM 3.1

If the assumptions (H1) – (H5) are satisfied and $x_0 \in X$, then the Cauchy problem (1) has a mild solution provided that

$$L_0 := L[M_0 + \frac{1}{\beta} C_{1-\beta} a^{\beta}] < 1 \qquad \text{-------------(5)}$$

$$(\gamma + \gamma_1 + M_3)M + M_0 L_1 + \frac{1}{\beta} C_{1-\beta} a^{\beta} L_1 < 1 \qquad \text{-------------(6)}$$

Where $M_0 = \| A^{-\beta} \|$.

Proof:

For the sake of brevity, we write that

$(t, x(t), x(b_1(t)), \ldots, x(b_m(t))) = (t, v(t))$

and $(t, x(t), x(a_1(t)), \ldots, x(a_m(t))) = (t, u(t))$.

Define the operator Q on E by the formula

$$(Qx)(t) = T(t)[x_0 + g(x)] - F(t, v(t)) - \int_0^t AT(t-s)F(s, v(s))ds + \int_0^t T(t-s)G(s, u(s))ds$$
$$+ \int_0^t T(t-s)K\left(s, x(s), \int_0^s k(s,\tau, x(\tau))d\tau, \int_0^a h(s,\tau, x(\tau))d\tau\right)ds$$

For positive integer r, let

$$B_r = \{x \in E : \| x(t) \| \le r, 0 \le t \le a\}$$

then for each r, Br is clearly a bounded closed convex set in E. Since by (2) and (3) the following relation holds:

$$\| AT(t-s)F(s, v(s)) \| \le \| A^{1-\beta} T(t-s) A^{\beta} F(s, v(s)) \|$$
$$\le \frac{C_{1-\beta}}{(t-s)^{1-\beta}} L_1 (r+1)$$

then from Bochners theorem (Marle, 1974) it follows that AT(t-s)F(s,v(s)) is integrable on [0,a], so Q is well defined on B_r.

Claim : there exists a positive integer r such that $QB_r \subseteq B_r$:

If it is not true, then for each positive integer r, there is a function $x_r(.) \in B_r$, but $Qx_r(t) \notin B_r$, that is $\|Qx_r(t)\| > r$ for some $t(r) \in [0,a]$, where t(r) denotes t is dependent of r. However, on the other hand, we have

$$r < \| Qx_r(t) \|$$
$$= \| T(t)[x_0 + g(x)] - F(t, v_r(t)) - \int_0^t AT(t-s)F(s, v_r(s))ds + \int_0^t T(t-s)G(s, u_r(s))ds$$
$$+ \int_0^t T(t-s)K\left(s, x(s), \int_0^s k(s,\tau, x(\tau))d\tau, \int_0^a h(s,\tau, x(\tau))d\tau\right)ds \ \|$$
$$\le M\| x_0 \| + MM_3 r + MM_4 + M_0 L_1 (r+1) + \frac{1}{\beta} C_{1-\beta} a^{\beta} L_1 (r+1) + M\int_0^t \phi(s)ds + M\int_0^t q(s)ds$$

Dividing on bothsides by r and taking the limit as $r \to \infty$, we get,

$$(\gamma + \gamma_1 + M_3)M + M_0 L_1 + \frac{1}{\beta} C_{1-\beta} a^{\beta} L_1 \ge 1$$

This contradicts (6). Hence for some positive integer r, $QB_r \subseteq B_r$.

Next we will show that the operator Q has a fixed point on B_r :

Let us decompose Q as $Q = Q_1 + Q_2$ where the operators Q_1 and Q_2 are defined on B_r respectively by

$$(Q_1 x)(t) = -F(t, v(t)) - \int_0^t AT(t-s)F(s, v(s))ds$$

$$(Q_2 x)(t) = T(t)[x_0 + g(x)] + \int_0^t T(t-s)G(s, u(s))ds$$
$$+ \int_0^t T(t-s)K\left(s, x(s), \int_0^s k(s,\tau, x(\tau))d\tau, \int_0^a h(s,\tau, x(\tau))d\tau\right)ds$$

for $0 \le t \le a$, and we will verify that Q_1 is contraction and Q_2 is a compact operator.

Claim : Q_1 is a contraction

Let $x_1, x_2 \in B_r$. Then for each $t \in [0,a]$ and by condition (H1) and (5), we have

$$\| (Q_1 x_1)(t) - (Q_1 x_2)(t) \| \le \| F(t, v_1(t)) - F(t, v_2(t)) \| + \left\| \int_0^t AT(t-s)[F(s, v_1(s)) - F(s, v_2(s))]ds \right\|$$
$$\le M_0 L \sup_{0\le s\le a} \| x_1(s) - x_2(s) \| + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} L ds \sup_{0\le s\le a} \| x_1(s) - x_2(s) \|$$
$$\le L\left[M_0 + \frac{1}{\beta} C_{1-\beta} a^{\beta}\right] \sup_{0\le s\le a} \| x_1(s) - x_2(s) \|$$
$$\le L_0 \sup_{0\le s\le a} \| x_1(s) - x_2(s) \|$$

Thus $\| Q_1 x_1 - Q_1 x_2 \| \le L_0 \| x_1 - x_2 \|$

So by assumption $0 < L_0 < 1$, we see that Q_1 is a contraction.

Claim : Q_2 is compact

To prove this we have to prove that Q_2 is continuous on B_r.

Let $\{x_n\} \subseteq B_r$ with $x_n \to x$ in B_r, then by (H2) (i), we have

$$G(s, u_n(s)) \to G(s, u(s)), \quad n \to \infty$$

$$K\left(t, x_n(t), \int_0^t k(t,s, x_n(s))ds, \int_0^a h(t,s, x_n(s))ds\right) \to K\left(t, x(t), \int_0^t k(t,s, x(s))ds, \int_0^a h(t,s, x(s))ds\right) \text{ as } n \to \infty$$

Since $\| G(s, u_n(s)) - G(s, u(s)) \| \le 2\phi_n(s)$,

$$\left\|K\left(t, x_n(t), \int_0^t k(t,s, x_n(s))ds, \int_0^a h(t,s, x_n(s))ds\right) - K\left(t, x(t), \int_0^t k(t,s, x(s))ds, \int_0^a h(t,s, x(s))ds\right)\right\| \le 2q_n(s)$$

by the dominated convergence theorem, we have

$$\| Q_2 x_n - Q_2 x \| = \sup_{0\le t\le a} \| T(t)[g(x_n) - g(x)] + \int_0^t T(t-s)[G(s, u_n(s)) - G(s, u(s))]ds$$
$$+ \int_0^t T(t-s)\left[K\left(s, x_n(s), \int_0^s k(s,\tau, x_n(\tau))d\tau, \int_0^a h(s,\tau, x_n(\tau))d\tau\right)\right.$$
$$\left. - K\left(s, x(s), \int_0^t k(s,\tau, x(\tau))d\tau, \int_0^a h(s,\tau, x(\tau))d\tau\right)\right]ds \|$$
$$\to 0 \text{ as } \quad n \to \infty$$

(i.e) Q_2 is continuous.

Next, we prove that $\{Q_2 x : x \in B_r\}$ is a family of equicontinuous functions. To see this we fix $t_1 > 0$ and $t_2 > t_1$ and $\varepsilon > 0$ be enough small. Then

$$
\begin{aligned}
\| (Q_2x)(t_2) - (Q_2x)(t_1) \| \le & \| T(t_2) - T(t_1) \| \| x_0 + g(x) \| + \int_0^{t_1-\varepsilon} \| T(t_2 - s) - T(t_1 - s) \| \| G(s, u(s)) \| ds \\
& + \int_{t_1-\varepsilon}^{t_1} \| T(t_2 - s) - T(t_1 - s) \| \| G(s, u(s)) \| ds \\
& + \int_{t_1}^{t_2} \| T(t_2 - s) \| \| G(s, u(s)) \| ds \\
& + \int_0^{t_1-\varepsilon} \| T(t_2 - s) - T(t_1 - s) \| \| K\left(s, x(s), \int_0^s k(s,\tau, x(\tau))d\tau, \int_0^a h(s,\tau, x(\tau))d\tau \right) \| ds \\
& + \int_{t_1-\varepsilon}^{t_1} \| T(t_2 - s) - T(t_1 - s) \| \| K\left(s, x(s), \int_0^s k(s,\tau, x(\tau))d\tau, \int_0^a h(s,\tau, x(\tau))d\tau \right) \| ds \\
& + \int_{t_1}^{t_2} \| T(t_2 - s) \| \| K\left(s, x(s), \int_0^s k(s,\tau, x(\tau))d\tau, \int_0^a h(s,\tau, x(\tau))d\tau \right) \| ds
\end{aligned}
$$

Note that $||G(s,u(s))|| \le \phi_n(s)$ and $\phi_n(s) \in L^1$, we see that $||Q_2x(t_2)-Q_2x(t_1)||$ tends to zero independently of $x \in B_r$ as $t_2-t_1 \to 0$. Since the compactness of T(t), t>0 implies the continuity of T(t), t>0 in t in the uniform operator topology.

We can prove that the function Q_2x, $x \in B_r$ are equicontinuous at t=0. Hence Q_2 maps B_r into a family of equicontinuous function.

Claim : $V(t)=\{(Q_2x)(t) : x \in B_r\}$ is relatively compact in X.

Let $0<t\le a$ be a fixed and $0 < \varepsilon < t$. For $x \in B_r$, we define

$$
\begin{aligned}
(Q_{2,\varepsilon}x)(t) &= T(t)[x_0 + g(x)] + \int_0^{t-\varepsilon} T(t-s)G(s, u(s))ds \\
&\quad + \int_0^{t-\varepsilon} T(t-s)K\left(s, x(s), \int_0^s k(s,\tau, x(\tau))d\tau, \int_0^a h(s,\tau, x(\tau))d\tau \right) ds \\
&= T(t)[x_0 + g(x)] + T(\varepsilon) \int_0^{t-\varepsilon} T(t-\varepsilon-s)G(s, u(s))ds \\
&\quad + T(\varepsilon) \int_0^{t-\varepsilon} T(t-\varepsilon-s)K\left(s, x(s), \int_0^s k(s,\tau, x(\tau))d\tau, \int_0^a h(s,\tau, x(\tau))d\tau \right) ds
\end{aligned}
$$

Then from compactness of $T(\varepsilon)$, $\varepsilon>0$ we obtain $V_\varepsilon=\{(Q_{2,\varepsilon}\, x)(t) : x \in B_r\}$is relatively compact in X for every ε, $0 < \varepsilon < t$. Moreover, for every $x \in B_r$, we have

$$
\begin{aligned}
\| (Q_2x)(t) - (Q_{2,\varepsilon}x)(t) \| &= \int_{t-\varepsilon}^{t} \| T(t-s)G(s, u(s)) \| ds \\
&\quad + \int_{t-\varepsilon}^{t} \| T(t-s)K\left(s, x(s), \int_0^t k(s,\tau, x(\tau))d\tau, \int_0^a h(s,\tau, x(\tau))d\tau \right) \| ds \\
&\le M \int_{t-\varepsilon}^{t} g_k(s)ds + M \int_{t-\varepsilon}^{t} q_r(s)ds
\end{aligned}
$$

Therefore, there are relatively compact sets arbitrarily close to the set V(t). Hence the set V(t) is also relatively compact in X.

Thus, by Arzela-Ascoli theorem, Q_2 is a compact operator. Those arguments enable us to conclude that $Q=Q_1+Q_2$ is a condensing map B_r, and by the Sadovskii's fixed point theorem there exist a fixed point x(.) for Q on B_r.Therefore, the Cauchy problem (1) has a mild solution, and the proof is completed.

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