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RESEARCH ARTICLE

PROJECTIVE TRANSFORMATION ON $N(k)$-CONTACT METRIC MANIFOLD

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Abstract

In this paper, we study projective vector fields on $N(k)$-contact metric manifolds. We obtain classification theorems for $N(k)$-contact metric manifolds admitting projective vector fields under the assumption of constant scalar curvature. we show that such a manifold is Sasakian or locally isometric to $E^{n+1}(0) \times S^n(4)$. A rigidity result is established showing that a projective flow is a curvature collineation under certain conditions.

Keywords: $N(k)$– contact metric manifold, Projective vector field, Sasakian, constant curvature, Ricci soliton.

1. Introduction

Contact metric geometry is an important area of differential geometry due to its deep connections with Riemannian geometry, topology and mathematical physics. A contact metric manifold is an odd dimensional manifold endowed with a structure (ϕ, ξ, η, g) satisfying certain compatibility conditions. The foundational study of such manifolds was developed by Blair [3].

A natural generalization of these manifolds is given by $N(k)$ -contact metric manifolds, introduced by Blair, Koufogiorgos, and Papantoniou [7] via the k-nullity distribution. These manifolds have been widely studied for their curvature properties and geometric structures [12]. Recent works further explore special structures such as Ricci solitons and Einstein-type conditions on these manifolds [8].

On the other hand, projective transformations preserve geodesics up to reparameterization and play a significant role in understanding geometric symmetries. The study of projective vector fields has been widely developed in the context of Riemannian and semi-Riemannian manifolds, where they are closely related to curvature tensors, especially the projective curvature tensor. Projective vector fields and curvature tensors have been extensively studied in various geometric settings, including contact metric manifolds [2]. Projective vector fields generalize several important classes of vector fields. In particular, every affine vector field 1 (for which $L_V\nabla = 0$) and every Killing vector field (which preserves the metric) is projective, but the converse is not true in general. Therefore, projective vector fields form a broader class that captures deeper geometric symmetries. Recent research has focused on under-standing projective symmetries in special geometric structures such as Sasakian manifolds and $N(k)$-contact metric manifolds [7,10]. In this context, projective vector fields provide an effective framework for analysing how geometric structures behave under transformations that preserve geodesic paths. Their study leads to important results concerning curvature restrictions, rigidity, and classification of manifolds.

Motivated by these developments, the present work investigates projective transformations on $N(k)$ -contact metric manifolds, focusing on the behaviour of the Levi-Civita connection and curvature tensor under projective vector fields.

2. PRELIMINARIES

A $(2n+1)$ -dimensional differentiable manifold M is said to admit an almost contact structure

if it admits a tensor field ϕ of type (1, 1), a vector field ξ, and a 1-form η satisfying [5,4]
$\phi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta \circ \phi = 0.$ (2.1)

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by

$J(X, f\, d/dt) = (\phi X - f\xi, \eta(X) d/dt)$ is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$.

Let g be a compatible Riemannian metric with the almost contact structure (ϕ, ξ, η)that is,

$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$ (2.2)

for all vector fields X, Y on M.

Then M becomes an almost contact metric manifold equipped with the almost contact metric structure (ϕ, ξ, η, g). In an almost contact metric manifold the following hold:

$\eta(X) = g(X, \xi), g(\phi X, Y) = -g(X, \phi Y),$ (2.3)

for all vector fields X, Y on M. An almost contact metric structure (ϕ, ξ, η, g) is said to be a contact metric structure if

$$d\eta(X, Y) = g(X, \phi Y), \quad (2.4)$$

for all vector fields X, Y on M. A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.5)$$

for all $X, Y \in TM$, where ∇ denotes the Levi-Civita connection of the Riemannian metric g. A contact metric manifold (M, ϕ, ξ, η, g) for which ξ is a Killing vector field is said to be a K-contact manifold. Blair, Koufogiorgos and Papantoniou [7] introduced the (k, μ)-nullity distribution on a contact metric manifold and gave several reasons for studying it. The (k, μ)-nullity distribution $N(k, \mu)$ of a contact metric manifold M is defined by

$$N(k, \mu): p \longmapsto N_p(k, \mu) = \{W \in T_p M : R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\},$$

for all $X, Y \in TM$, where $\xi \in N(k)$ and R is Riemann curvature tensor.

A contact metric manifold M with $\xi \in N(k, \mu)$ is called a (k, μ)-contact metric manifold. If $\mu = 0$, the (k, μ)-nullity distribution reduces to the k-nullity distribution.

The k-nullity distribution $N(k)$ of a Riemannian manifold is defined by [9]

$$N(k): p \longmapsto N_p(k) = \{Z \in T_p M : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\}$$

where k is a constant. If the characteristic vector field $\xi \in N(k)$, then the contact metric manifold is called an $N(k)$-contact metric manifold [1]. If $k = 1$, then the manifold is Sasakian, and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [4].

However, for an $N(k)$- contact metric manifold M of dimension $(2n + 1)$, we have [6]

$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX$ (2.6)

$((\nabla_X \eta)(Y) = -g(\phi X + \phi hX, Y),)$ (2.7)

$(\nabla_X \xi = -\phi X - \phi hX,)$ (2.8)

where $h = \frac{1}{2} L_\xi \phi$.

Moreover, $h^2 = (k - 1)\phi^2$. In an $N(k)$ - contact metric manifold the following hold:

$(R(X, Y)\xi \& = k[\eta(Y)X - \eta(X)Y],)$ (2.9)

$(R(\xi, X)Y \& = k[g(X, Y)\xi - \eta(Y)X],)$ (2.10)

$$S(Y, \xi) = 2nk\eta(Y), \quad (2.11)$$

where S denotes the Ricci tensor.

Definition 2.1. A vector field V on a Riemannian manifold M with connection ∇ is called a projective vector field if there is a scalar 1-form P on M satisfying

$$(L_V \nabla)(X, Y) = p(X)Y + p(Y)X. \quad (2.12)$$

It is to be noted that if $L_V \nabla = 0$, V is called affine vector field and V is Killing if $L_V g = 0$. A Vvector field is torsforming on a $N(k)$-contact metric manifold if

$$\nabla_X V = \rho X + \eta(X)V, \quad (2.13)$$

where ρ is a scalar function.

From (2.13) and $\nabla_X \xi = -\phi X - \phi hX$, we compute

$L_V \xi = \nabla_V \xi - \nabla_\xi V$, we have

$$L_V \xi = -\phi V - \phi hV - \rho\xi - V. \quad (2.14)$$

From (2.13) again, we get

$$(L_V g)(X,\xi) \quad = g(\nabla_X V,\xi) + g(X,\nabla_\xi V)$$

$$(L_V g)(X,\xi) \quad = g(\rho X + \eta(X)V,\xi) + g(X,\rho\xi + V)$$

$(L\_V\ g)(X,\xi) = 2\rho\eta(X) + \eta(X)\eta(V) + g(X,V)$ (2.15)

$((L\_V\ \eta)(X)\& = (L\_V\ g)(X,\xi) + g(X,L\_V\ \xi)@\&$
$= 2\rho\eta(X) + \eta(X)\eta(V) + g(X,V) + g(X,-\phi V - \phi hV - \rho\xi - V))$

$= \rho\eta(X) + \eta(X)\eta(V) - g(X,\phi V) - g(X,\phi hV)$. (2.16)

$$(L_V\phi)(X) \quad = L_V(\phi X) - \phi(L_V X)$$
$$= \nabla_V(\phi X) - \nabla_{\phi X}V - \phi(\nabla_V X - \nabla_X V)$$
$$= (\nabla_V\phi)X - \nabla_{\phi X}V + \phi\nabla_X V$$
$$= g(X,V)\xi + g(hV,X)\xi - \eta(X)V - \eta(X)hV$$
$$-\rho\phi X + \rho\phi X + \eta(X)\phi V$$

$= g(X,V)\xi + g(hV,X)\xi - \eta(X)V - \eta(X)hV + \eta(X)\phi V$. (2.17)

Thus we have

Theorem 2.2. In an $N(k)$-contact metric manifold, a projective vector field V which is also a torse forming vector field transforms the structure tensors ϕ,ξ,η and the metric g according as (2.14),(2.16), (2.17) and (2.15) respectively.

Proof:

We have the following commutation formula due to Yano[11]

$((L\_V\ R)(X,Y)Z\& = (\nabla\_X\ L\_V\ \nabla)(Y,Z) - (\nabla\_Y\ L\_V\ \nabla)(X,Z).)$
(2.18)

Using (2.12) in the above equation, we get

$(L\_V\ R)(X,Y)Z = (\nabla\_X\ p)(Y)Z - (\nabla\_Y\ p)(X)Z + (\nabla\_X\ p)(Z)Y - (\nabla\_Y\ p)(Z)X.$
(2.19)

Setting $Z =$ ξ in the above equation, we obtain

$(L\_V\ R)(X,Y)\xi = (\nabla\_X\ p)(Y)\xi - (\nabla\_Y\ p)(X)\xi + (\nabla\_X\ p)(\xi)Y - (\nabla\_Y\ p)(\xi)X.$
(2.20)

3. Projective vector fields on $N(k)$-Contact metric manifolds

In this section we consider V as a projective vector field which is also conformal on a $(2n+1)$ dimensional $N(k)$ contact metric manifold M.

As is well known that a vector field V is a conformal vector field if $\nabla_X V = fX$, where f is a scalar function.

Suppose the projective vector field V is also conformal. i.e,

$$\nabla_X V = fX \tag{3.1}$$

and $(L_V\nabla)(X,Y) = p(X)Y + p(Y)X$,

where p is a 1-form.

We now compute

$$L_V\xi = [V,\xi] = \nabla_V\xi - \nabla_\xi V. \tag{3.2}$$

From (2.8) and (3.1) in the above equation, we get

$$L_V\xi = -\phi V - \phi hV - f\xi. \tag{3.3}$$

Taking Lie derivative of (2.9),with respect to V, we obtain

$(L\_V\ R)(X,Y)\xi = (L\_V\ \eta)(X)Y - (L\_V\ \eta)(Y)X - R(X,Y)L\_V\ \xi.$ (3.4)

Taking $Y = \xi$ and using (2.9), we obtain

$(L\_V\ R)(X,Y)\xi = (L\_V\ \eta)(X)Y - (L\_V\ \eta)(Y)X - R(X,Y)L\_V\ \xi.$ (3.5)

Using (3.1) and (2.9), we obtain

$(L\_V\ R)(X,\xi)\xi = (L\_V\ \eta)(X)\xi - (L\_V\ X)(\xi)X - R(X,\xi)\ L\_V\ \xi.$ (3.6)

Using (3.3) in the above equation, we get

$R(X,\xi)L\_V\ \xi = k[-fX + g(X,\phi V + \phi hV + f\xi)\xi]$
(3.7)

From (3.5) contracting with ξ, we get

$(g((L\_V\ R)(X,\xi)\xi,\xi)\& = (L\_V\ \eta)(X) - (L\_V\ \eta)(\xi)\eta(X) + k(f\eta(X) - g(X,\phi V)@\& - g(X,\phi hV) - f\eta(X)).)$ (3.8)

Hence,

$$(L_V\eta)(X) = f\eta(X). \tag{3.9}$$

Therefore V is a contact transformation.Now,

$$(L_V\phi)(X) \quad = L_V(\phi X) - \phi(L_V X)$$
$$= \nabla_V(\phi X) - \nabla_{\phi X}V - \phi(\nabla_V X) + \phi(\nabla_X V)$$

$$= (\nabla_V\phi)(X) - \nabla_{\phi X}V + \phi(\nabla_X V). \tag{3.10}$$

Using (2.6) and (3.3) in (3.10), we have

$$(L_V\phi)(X) \quad = g(V + hV, X)\xi - \eta((V + hV). \tag{3.11}$$

Also from $\nabla_X V = fX$ and $(L_V g)(X,Y) = g(\nabla_X V,Y) + g(X,\nabla_Y V)$, we have

$$(L_V g)(X,Y) \quad = 2fg(X,Y). \tag{3.12}$$

Since d commutes with L_V and $d\eta(X,Y) = g(X,\phi Y)$ we have.

$(dL_V\eta))(X,Y) = (L_V g)(X,\phi Y) + g(X,(L_V\phi)Y)$ (3.13)

But $(L_V\eta)(X) = f\eta(X)$ in $(dL_V\eta)(X,Y) = X(L_V\eta)(Y) - Y(L_V\eta)(X) - (L_V\eta)[X,Y]$ gives.

$(dL_V\eta)(X,Y) = (Xf)\eta(Y) - (Yf)\eta(X) + f((\nabla_X\eta)(Y) - (\nabla_Y\eta)(X))$ (3.14)

Using (3.10), (3.11) and (3.13) in (3.12), we get

$$(Xf)\eta(Y) + (Yf)\eta(X) - 2fg(\phi hY, X) = 2fg(X, \phi Y) + g(V + hV, X)\eta(Y) - g(V + hV, Y)\eta(X). \quad (3.15)$$

Tracing with respect to X and Y gives

$$\xi f = 0. \quad (3.16)$$

Putting $Y = \xi$ in (3.14), we obtain

$$Xf = g(V + hV, X) - \eta(V)\eta(X), \quad (3.17)$$

or

$$grad f = V + hV - \eta(V)\xi. \quad (3.18)$$

Using (3.3) and (2.9) in (3.1), we have

$$(L_V R)(X,Y)\xi = f(\eta(Y)X - \eta(X)Y) + R(X,Y)\phi V + R(X,Y)\phi hV - f(\eta(Y)X - \eta(X)Y)$$

$$= R(X,Y)\phi V + R(X,Y)\phi hV \quad (3.19)$$

we have

$$(L_V S)(Y,Z) = (\nabla_Z p)(Y) - (2n+1)(\nabla_Y p)(Z) \quad (3.20)$$

From (2.12) and (3.11) and we have the formula

$(\nabla_X L_V g)(Y,Z) = g((L_V \nabla)(X,Y), Z) + g((L_V \nabla)(X,Z), Y)$, we have

$$2p(X)g(Y,Z) + p(Y)g(X,Z) + p(Z)g(X,Y) = 2(Xf)g(Y,Z) \quad (3.21)$$

Putting $X = Z = e_i$ in(3.20), we obtain

$$p(Y) = \frac{1}{n+2} Yf \quad (3.22)$$

Substituting this in (2.19) and using (3.18), $\xi f = 0$,

$$R(X,Y)\phi V + R(X,Y)\phi hV = (\phi X(f) + \phi hX(f))Y \quad (3.23)$$

Taking $Y = \xi$ and using (2.9) in the above equation, we get

$$-k(\eta(X)(\phi V + \phi hV)) = (\phi Xf + \phi hXf)\xi. \quad (3.24)$$

Contracting (3.22) with Z and taking

$$g(R(X,Y)\phi V, Z) + g(R(X,Y)\phi hV, Z) = (\phi X(f) + \phi hX(f))g(Y,Z) - ((\phi Y)f + (\phi hY)f)g(X,Z). \quad (3.25)$$

Contracting (3.23) with ξ, we get

$$(\phi X + \phi hX)f = 0. \quad (3.26)$$

Now changing h to hX and ϕ to ϕX in the above equation, we have This implies

$$(-hX + (1-k)X)f = 0$$

$$h(grad f) = (1-k) grad f. \quad (3.27)$$

Since $(n+2)p(Y) = Yf$, if $p(X) = g(X,U)$ then, we get $grad f = (n+2)U$.

This value of $grad f$ in (3.26) gives

$$hU = (1-k)U. \quad (3.28)$$

If h annihilates U, then $k = 1$ and M becomes Sasakian. Thus from (3.27), we have the following.

Theorem 3.1. Let V be a projective vector field which is also conformal on $N(k)$-manifold M. Then vector field U corresponding to one form p is an eigenvector of h with eigen value $(1-k)$. Further if h annihilates U then M is Sasakian.

Proof. From (3.21), we have

$$p(X) = df_1(X), \quad (3.29)$$

where $f_1 = \frac{1}{n+2} f$.

From (2.19) and (3.29), we derive

$$(2n+1)(L_V R)(X,Y)Z = Hessf_1((X,Z)Y - Hessf_1((Y,Z)X \quad (3.30)$$

where $Hessf(X,Y) = g(\nabla_X grad f Y)$,

$$(L_V S)(X,Y) = \left(\frac{1}{2n+1}\right)(Hessf((X,Y) - \Delta f g(X,Y)), \quad (3.31)$$

where $\Delta f = div grad f$ is the Laplacian of f.

Thus from (3.30), we have

Theorem 3.2. A projective conformal vector field V on a $N(k)$ -manifold is a curvature collineation if and only if $Hessf(X,Z)Y = Hessf(Y,Z)X$ and Ricci collineation if and only if $Hessf((X,Y) = \Delta f g(X,Y)$.

Proof:

Further if $Hessf(X,Y) = \lambda_1 g(X,Y)$, then from (3.29), we have

$$(L_V R)(X,Y)Z = \lambda(g(X,Z)Y - g(Y,Z)X), \quad (3.32)$$

where $\lambda = \frac{\lambda_1}{2n+1}$,

Thus we have

Theorem 3.3. A projective vector field on $N(k)$-manifold M which is also conformal trans-forms M into manifold of constant curvature like structure.

Proof:

Further contraction of (3.31) with respect to X and Y gives
$L_V r = -2n\nabla f$.

If g is a Ricci soliton with potential vector field V as projective vector field, then from the Ricci soliton equation
$(L_V g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z)$ and $p(X) = \frac{1}{2n+1} Xf$ in the commutation formula
$(\nabla_X L_V g)(Y,Z) = g((L_V\nabla)(X,Y),Z) + g((L_V\nabla)(X,Z),Y)$,

we have

$$-2(n+1)(\nabla_X S)(Y,Z) = 2(Xf)g(Y,Z) + (Yf)g(X,Z) + (Zf)g(X,Y),$$

which then implies that

$$Xr = (\nabla_X r) = -\frac{(n-1)}{(n+1)} Xf.$$

Therefore the scalar curvature r is constant if and only if f is constant. □

Thuse we have

Theorem 3.4. Let V be a projective vector field on $N(k)$ -manifold M which is also conformal with conformal factor f. Then scalar curvature r is constant if and only if f is a constant.

Proof.

Using
$(L_V\nabla)(X,Y) = \nabla_X\nabla_Y V - \nabla_{\nabla_X Y} V + R(V,X)Y$
and (2.12), we have

$$R(V,X)Y = p(X)Y + p(Y)X - (Xf)Y \qquad (3.33)$$

Applying η on both sides of the above equation, we get

$$\eta(R(V,X)Y) = p(X)\eta(Y) + p(Y)\eta(X) - (Xf)\eta(Y). \qquad (3.34)$$

On the other hand, contracting (2.9) with V, we have

$$\eta(R(X,Y)V) = k[\eta(Y)g(X,V) - \eta(X)g(Y,V)] \qquad (3.35)$$

Comparing the two expressions, we obtain

$$p(X)\eta(Y) + p(Y)\eta(X) - (\nabla_X p)\eta(Y) = k[\eta(Y)g(X,V) - \eta(X)g(Y,V)]. \qquad (3.36)$$

Setting $Y = \xi$ in (3.35), we get

$$p(X) + p(\xi)\eta(X) - Xf = k[g(X,V) - \eta(X)\eta(V)] \qquad (3.37)$$

Taking $X = \xi$ in (3.36), we get $p(\xi) = \xi f = 0$. Therefore

$$P(X) - (df)X = k[g(X,V) - \eta(X)\eta(V)]. \qquad (3.38)$$

Since $P(X) = \frac{1}{2n+1} Xf = \frac{1}{2n+1} df(X)$ gives

$$\frac{2n}{2n+1} df(X) = k[g(X,V) - \eta(X)\eta(V)]$$

Thus from Theorem (3.4), we have

Theorem 3.5. If $N(k)$ -manifold M admitting conformal projective vector field is of constant curvature, then it is isometric to $E^{n+1}(0) \times S^n(4)$.

4. CONCLUSION

In this paper, we have investigated projective vector fields on $N(k)$- contact metric manifolds and established classification and rigidity results. We showed that, under suitable conditions, the manifold becomes Sasakian or is locally isometric to $E^{n+1}(0) \times S^n(4)$. Furthermore, projective conformal vector fields induce curvature and Ricci collineations under appropriate conditions, while the scalar curvature is constant if and only if the conformal factor is

constant. These results demonstrate that projective transformations impose strong geometric restrictions on $N(k)$ -contact metric manifolds and reveal significant rigidity phenomena in contact metric geometry.

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