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RESEARCH ARTICLE

STAR DECOMPOSITION OF SOME COMPLETE MULTIPARTITE GRAPHS

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Abstract

Let G be a graph with no isolated vertices, and consider a subgraph H of G. Then the graph G admits an H-decomposition, if its edge set E(G) can be partitioned into pair wise edge-disjoint subgraphs, each isomorphic to H . Let S_n represent a star with n vertices. In this paper, complete four-partite graphs that can be decomposed into the union, $S_3 \cup S_4$ are characterised.

Keywords: Decomposition, Complete Four-Partite Graph, Star

1. INTRODUCTION

Finite and undirected graphs are considered in this paper. The complete four-partite graph is denoted by $K_{k,l,m,n}$ and the star $K_{1,n-1}$ by S_n .If the star S_n has vertex set $\{v_1,v_2,v_3,\ldots,v_n\}$ with edge set $\{v_1v_i: i=2,3,\ldots,n\}$, it may alternatively be written as as $S\{v_1:v_2,v_3,\ldots,v_n\}$.

A characterization of bipartite graphs that admits a decomposition into copies of the union of two arbitrary stars S_p and S_q is given in [1]. In this work, we focus on the specific case p = 3 and q = 4 and attempt the decomposition of four-partite graphs. The case of p=2 and q=3 is given in [2].

2. PRELI MINARIES

A graph G is *m-partite*, m>1, if it is possible to partition V(G) into subsets or partite sets V_1, $V_2 \ldots V_m$, such that every edge of G joins a vertex of V_i to a vertex of V_j for $i \neq j$. In a complete m-partite graph, every vertex u in V_i is adjacent to each v in V_j for $i \neq j$.

Let G be any graph and $F=\{G_1,G_2,\ldots G_k\}$ be a family of subgraphs of G. An F -decomposition of G is an edge-disjoint decomposition of G into copies of G_i for i = 1,2,...k, provided, no G_i has isolated vertices. Obviously, k_i is a positive integer. If each G_i is isomorphic to a graph H then G is H- decomposable or G has an H-decomposition, denoted by $H \mid G$. Clearly, when $H \mid G$,then $e(H) \mid e(G)$, but not conversely.

3. DECOMPOSITION OF $K_{k,l,m,n}$, THE COMPLETE FOUR-PARTITE GRAPH INTO $S_3 \cup S_4$

3.1. Lemma: If $H \mid G_1$ and $H \mid G_2$ then $H \mid G_1 \cup G_2$.

3.2 Theorem [1] Let m, n, p, q be positive integers such that m,n ≥ 2 and p ≠ q. The necessary and sufficient condition that the complete bipartite graph $K_{m,n}$ is $S_{p+1} \cup S_{q+1}$ -decomposable is (i) $m+n \geq p+q+2$ and (ii) $mn \equiv 0(\mathrm{mod}(p+q))$.

3.3 Theorem [1] The complete tripartite graph $G = K_{l,m,n}$ is $S_3 \cup S_4$ decomposable if and only if $e(G) \equiv 0(\mathrm{mod}\,5)$, $l+m+n \geq 7$, $G \neq K_{1,3,3}$ and $G \neq K_{1,1,5n+2}$, $n \geq 1$.

Proof: When $G = K_{l,m,n}$ is $S_3 \cup S_4$ -decomposable, necessarily we have, $e(G) \equiv 0(\mathrm{mod}\ 5)$.The stars S_3 and S_4 are vertex disjoint. Hence we have, $l + m + n \geq 7$. If $G = K_{1,1,5n+2}$, $e(G) = 10n+15 \equiv 0(\mathrm{mod}5)$. Let $V(K_{1,1,5n+2}) = \{u,v,w_1,w_2,\ldots w_{5n+2}\}$.In any $S_3 \cup S_4$decomposition of G , the edge uv cannot be included in any copy of $S_3 \cup S_4$ as both the vertices are adjacent to all the remaining vertices. Conversely, when $e(G) \equiv 0(\mathrm{mod}\ 5)$, $lm + mn + ln \equiv 0(\mathrm{mod}\ 5)$, which is true when

at least two of l, m, n are multiples of 5
$l \equiv 1(\mathrm{mod}\ 5)$,$m \equiv 3(\mathrm{mod}\ 5)$ and $n \equiv 3(\mathrm{mod}\ 5)$
$l \equiv 1(\mathrm{mod}\ 5)$,$m \equiv 1(\mathrm{mod}\ 5)$ and $n \equiv 2(\mathrm{mod}\ 5)$
$l \equiv 2(\mathrm{mod}\ 5)$,$m \equiv 2(\mathrm{mod}\ 5)$ and $n \equiv 4(\mathrm{mod}\ 5)$
$l \equiv 3(\mathrm{mod}\ 5)$,$m \equiv 4(\mathrm{mod}\ 5)$ and $n \equiv 4(\mathrm{mod}\ 5)$

Case (i): Let $l = 5x$, $m = 5y$ and $n = z$, where, x, y, z ≥ 1.
$E(K_{l,m,n}) = E(K_{5x,5y,z}) = E(K_{5x,5y+z}) \cup E(K_{5y,z})$

By Theorem 3.2, the complete bipartite graphs on the right hand side of the above equality are $S_3\cup S_4$ decomposable and lemma 3.1 proves the $S_3\cup S_4$ -decomposability $K_{l,m,n}$.

Case (ii) Let k=5a, l = 5b+1 ,m = 5c+3 and n = 5d+3.The least values of l, m, n are 1,3,3; but the graph $K_{1,3,3}$ is not $S_3\cup S_4$ – decomposable. Hence we choose, x ,y $\geq$ 0 and z $\geq$ 1.

$E\ (K_{l,m,n}) = E(K_{5x+1,5y+3,5z+2}) = E(K_{5x,5y+5z+6}) \cup E(K_{1,5y+3,5z+3})$

$= E(K_{5x,5y+5z+6,}) \cup E(K_{5y,5z'+9}) \cup E(K_{1,3,5z'+8})$ where $z' = z-1$

$= E(K_{5x,5y+5z+6,}) \cup E(K_{5y,5z'+9}) \cup E(K_{5z',9}) \cup E(K_{1,3,8})$

Applying Lemma 3.1, Theorem 3.2 and decomposition 3.3.4 in the last equation, we see that, $K_{l,m,n}$ is $S_3 \cup S_4$ -decomposable.

Case (iii) Let k=5a, l = 5b +1 , m = 5c +1 and n = 5d +2, where, a,b,c,d $\geq$ 1

$E\ (K_{l,m,n}) = E(K_{5x+1,5y+1,5z+2}) = E(K_{5x,5y+5z+3}) \cup E(K_{1,5y+1,5z+2})$

$E(K_{1,5y+1,5z+2}) = E(K_{5y+2,5z}) \cup E(K_{1,5y+1,2})$

$E(K_{1,5y+1,2}) = E(K_{5(y-1),3}) \cup K_{1,2,6}$

Applying Lemma 3.1, Theorem 3.2 and decomposotion 3.3.3 to the last equation and moving backwards, we get the $S_3 \cup S_4$ decomposability of E ($K_{l,m,n}$)

Case (iv) Let l = 5x +2, m = 5y+2 and n = 5z+4, where, x, y, z $\geq$ 1

$E\ (K_{l,m,n}) = E\ (K_{5x+2,5y+2,5z+4}) = E\ (K_{5x,5y+5z+6}) \cup E\ (K_{2,5y+2,5z+4})$

$= E(K_{5x,5y+5z+6}) \cup E(K_{5y,5z+6}) \cup E(K_{2,2,5z+4})$

$= E(K_{5x,5y+5z+6}) \cup E(K_{5y,5z+6}) \cup E(K_{5z,4}) E(K_{2,2,4})$

$E\ (K_{2,2,4})$ is $S_3\cup S_4$ - decomposable. Hence the proof as in case (iii)

Case (v) Let l = 5x +3, m = 5y +4 and n = 5z +4 where, x, y, z $\geq$ 1

$E(K_{l,m,n}) = E(K_{5x+3,5y+4,5z+4}) = E(K_{5x,5y+5z+8}) \cup E(K_{3,5y+4,5z+4})$

$= E(K_{5x,5y+5z+8}) \cup E(K_{5y,5z+7}) \cup E(K_{4,3,5z+4})$

$= E(K_{5x,5y+5z+8}) \cup E(K_{5y,5z+7}) \cup E(K_{5z,,7}) \cup E(K_{3,4,4})$,

Where $(K_{3,4,4})$is $S_3\cup S_4$ - decomposable, which completes the proof. Next, we examine the $S_3\cup S_4$ decomposability of the complete four-partite graphs. The following elementary decompositions are used in the proof.

3.3.1 Decomposition of $K_{1,1,1,9}$

Let V($K_{1,1,1,9}$) = {11}∪ {21}{31}∪{41,42,43,44,45,46,47,48,49}.The partition is S{11:21,42} ∪{31;41,43,48}, S{21:42,49} ∪{11;31,43,44}, S{31:21,44} ∪{11;45,46,47}, S{41:11,21} ∪{31;45,46,47},S{31:42,49} ∪{21;43,44,48},S{11:48,49} ∪{21;45,46,47}

3.3.2 Decomposition of K $_{1,2,3,4}$

Let V($K_{1,2,3,4,}$) ={11}∪ {21,22} ∪ {31,32,33}∪ {41,42,,43,44}.The edges are decomposed as, S{11:41,31,32}∪{44;21,33}, S{11:21,23,42}∪{22;31,32}, S{11:22,43,44}∪S{31;41,42}, S{22:33,41,42}∪{43;31,21}, S{21:33,41,42}∪ {22;43,44}, S{32:41,42,43}∪ {21;31,33}, S{44:31,32}∪ S{33;41,42,43}.

3.3.3 Decomposition of K $_{1,4,4,4}$

Let V($K_{1,4,4,4,}$) ={11}∪ {21,22,23,24} ∪ {31,32,33,34}∪ {41,42,,43,44}. K $_{1,4,4,4}$ has the following $S_3\cup S_4$ – decomposition. S{24:43,44}∪ S{11;21,22,23}, S{23:41,42}∪ S{11;31,32,24}, S{24:41,42}∪ S{11;41,34,33}, S{24:31,32}∪ S{11;44,42,43}, S{24:33,34}∪ S{21;31,32,41}, S{22:31,32}∪ S{22;44,42,43}, S{23:43,44}∪ S{22;41,33,34}, S{23:33,34}∪ S{22;42,43,44}, S{21:33,34}∪ S{31;41,42,43}, S{44:33,34}∪ S{32;41,42,43}, S{44:31,32}∪ S{33;41,42,43}, S{23:31,32}∪ S{34;41,42,43}.

3.3.4 Decomposition of K $_{2,2,2,3}$

Let V($K_{2,3,3,3,}$) ={11,12}∪ {21,22,23} ∪ {31,32,33}∪ {41,42,43}.The $S_3\cup S_4$ decomposition of this graph is S{21:11,32}∪ S{31;41,42,43}, S{12:41,42}∪ S{11;22,31,43}, S{21:31,43}∪ S{11;41,42,32}, S{21:41,42}∪ S{12;31,32,43}, S{12:21,22}∪ S{32;41,42,43}.

3.3.5 Decomposition of K $_{2,3,3,3}$

Let V($K_{2,3,3,3}$) ={11,12}∪ {21,22,23} ∪ {31,32,33}∪ {41,42,43}. The **$S_3\cup S_4$** decomposition is S{22:42,43 }∪ S{11;21,22,23}, S{12:31,41}∪ S{11;31,32,33}, S{12:32,33}∪ S{11;41,42,43}, S{31:41,42}∪ S{22;31,32,33}, S{22:41,42}∪ S{21:32,33,43}, S{22:33,43}∪ S{21;31,41,42}, S{32:41,42}∪ S{23;31,33,43}, S{43:31,32}∪ S{33;41,42,33}.

Now we prove the main result.

3.4 Theorem

The complete 4-partite graph G = $K_{k,l,m,n}$, $k+l+m+n \geq 7$ is$S_3\cup S_4$ - decomposable if and only if e(G) ≡ 0 (mod5) and G ≠ K $_{1,1,1,5n+4}$

Proof: When G is decomposable, it necessarily satisfies e(G) ≡ 0 (mod5). Also we have, $k+l+m+n \geq 7$, for$S_3$and S_4 are vertex disjoint.

Sufficiency: If e(G) ≡ 0(mod5), we have

at least three of k, l , m, n ≡ 0(mod5)

k ≡ 0(mod5), l ≡ 1(mod5) , m ≡ 1(mod5) and n ≡ 2(mod5)

k ≡ 0(mod5), l ≡ 1(mod5) , m ≡ 3(mod5) and n ≡ 3(mod5)

$k \equiv 0(\text{mod}5)$, $l \equiv 2(\text{mod}5)$, $m \equiv 2(\text{mod}5)$ and $n \equiv 4(\text{mod}5)$
$k \equiv 0(\text{mod}5)$, $l \equiv 3(\text{mod}5)$, $m \equiv 4(\text{mod}5)$ and $n \equiv 4(\text{mod}5)$
$k \equiv 1(\text{mod}5)$, $l \equiv 1(\text{mod}5)$, $m \equiv 1(\text{mod}5)$ and $n \equiv 4(\text{mod}5)$
$k \equiv 1(\text{mod}5)$, $l \equiv 2(\text{mod}5)$, $m \equiv 3(\text{mod}5)$ and $n \equiv 4(\text{mod}5)$
$k \equiv 1(\text{mod}5)$, $l \equiv 4(\text{mod}5)$, $m \equiv 4(\text{mod}5)$ and $n \equiv 4(\text{mod}5)$
$k \equiv 2(\text{mod}5)$, $l \equiv 2(\text{mod}5)$, $m \equiv 2(\text{mod}5)$ and $n \equiv 3(\text{mod}5)$
$k \equiv 2(\text{mod}5)$, $l \equiv 3(\text{mod}5)$, $m \equiv 3(\text{mod}5)$ and $n \equiv 3(\text{mod}5)$

Case(i): Let k = 5a, l = 5b, m = 5c and n>1
$E(K_{k,l,m,n}) = E(K_{5a,5b,5c,n}) = E(K_{5a,5b+5c+n}) \cup E(K_{5b,5c,n})$
$= E(K_{5a,5b+5c+n}) \cup E(K_{5b,5c+n}) \cup E(K_{5c,n})$ …..(*)

When n =1,
$E(K_{5a,5b,5c,1}) = E(K_{5a,5b+5c+1}) \cup E(K_{5b,5c,1})$
$= E(K_{5a,5b+5c+1}) \cup E(K_{5(b-1),5c+1}) \cup E(K_{5,5c+1})$
$= E(K_{5a,5b+5c+1}) \cup E(K_{5(b-1),5c+1}) \cup E(K_{5(c-1),6}) \cup E(K_{5,5,1})$

Case(ii): Let k = 5a, l = 5b+1, m = 5c+1 and n=5d+2, where a ≥ 0, b,c,d ≥ 1
$E(K_{k,l,m,n}) = E(K_{5a,5b+1,5c+1,5d+2}) = E(K_{5a,5b+5c+5d+4}) \cup E(K_{5b+1,5c+1,5d+2})$
$= E(K_{5a,5a,5b+5c+5d+4}) \cup E(K_{5b,5c+5d+3}) \cup E(K_{1,5c+1,5d+2})$
$= E(K_{5a,5a,5b+5c+5d+4}) \cup E(K_{5b,5c+5d+3}) \cup E(K_{5c+2,5d}) \cup E(K_{1,5c+1,2})$
$= E(K_{5a,5a,5b+5c+5d+4}) \cup E(K_{5b,5c+5d+3}) \cup E(K_{5c+2,5d}) \cup E(K_{5(c-1),3}) \cup K_{1,2,6}$

Case(iii): Let k = 5a, l = 5b+1, m = 5c+3 and n=5d+3, where a,b,c ≥ 0 and d ≥ 1

$E(K_{k,l,m,n}) = E(K_{5a,5b+1,5c+3,5d+3}) = E(K_{5a,5b+5c+5d+7}) \cup E(K_{5b+1,5c+3,5d+3})$
$= E(K_{5a,5b+5c+5d+7}) \cup E(K_{5b,5c+5d+6}) \cup E(K_{1,5c+3,5d+3})$
$= E(K_{5a,5b+5c+5d+7}) \cup E(K_{5b,5c+5d+6}) \cup E(K_{5c,5d'+9}) \cup E(K_{1,3,5d'+8})$ where $d' = d - 1$
$= E(K_{5a,5b+5c+5d+7}) \cup E(K_{5b,5c+5d+6}) \cup E(K_{5c,5d'+9}) \cup E(K_{5d',9}) \cup E(K_{1,3,8})$

Case(iv): Let k = 5a, l = 5b+2, m = 5c+2 and n=5d+4, where a ≥ 0 and b,c,d ≥ 1
$E(K_{k,l,m,n}) = E(K_{5a,5b+2,5c+2,5d+4}) = E(K_{5a,5b+5c+5d+8}) \cup E(K_{5b+2,5c+2,5d+4})$
$= E(K_{5a,5b+5c+5d+8}) \cup E(K_{5b,5c+5d+6}) \cup E(K_{2,5c+2,5d+4})$
$= E(K_{5a,5b+5c+5d+8}) \cup E(K_{5b,5c+5d+6}) \cup E(K_{5c,5d+6}) \cup E(K_{2,2,5d+4})$
$= E(K_{5a,5b+5c+5d+8}) \cup E(K_{5b,5c+5d+6}) \cup E(K_{5c,5d+6}) \cup E(K_{5d,4}) E(K_{2,2,4})$

Case (v) : Let k= 5a, l= 5b+3, m= 5c+4 and n= 5d+4
$E(K_{k,l,m,n}) = E(K_{5a,5b+3,5c+4,5d+4}) = E(K_{5a,5b+5c+5d+11}) \cup E(K_{5b+3,5c+4,5d+4})$
$= E(K_{5a,5b+5c+5d+11}) \cup E(K_{5b,5c+5d+8}) \cup E(K_{3,5c+4,5d+4})$
$= E(K_{5a,5b+5c+5d+11}) \cup E(K_{5b,5c+5d+8}) \cup E(K_{5c,5d+7}) \cup E(K_{4,3,5d+4})$
$= E(K_{5a,5b+5c+5d+11}) \cup E(K_{5b,5c+5d+8}) \cup E(K_{5c,5d+7}) \cup E(K_{5d,,7}) \cup E(K_{3,4,4})$,

Case (vi): Let k= 5a+1, l= 5b+1, m= 5c+1 and n= 5d+4
$E(K_{k,l,m,n}) = E(K_{5a+1,5b+1,5c+1,5d+4})$
$= E(K_{5a,5b+5c,5d+6}) \cup E(K_{1,5b+1,5c+1,5d+4})$
$= E(K_{5a,5b+5c,5d+6}) \cup E(K_{5b,5c+5d+6}) \cup E(K_{1,1,5c+1,5d+4})$
$= E(K_{5a,5b+5c,5d+6}) \cup E(K_{5b,5c+5d+6}) \cup E(K_{5c5d+6,}) \cup E(K_{1,1,1,5d+4,})$
$= E(K_{5a,5b+5c,5d+6}) \cup E(K_{5b,5c+5d+6}) \cup E(K_{5c5d+6,}) \cup E(K_{5(d-1),3}) \cup E(K_{1,1,1,9,})$

Case (vii):Let k= 5a+1, l= 5b+2, m= 5c+3 and n= 5d+4
$E(K_{5a+1,5b+2,5c+3,5d+4}) = E(K_{5a,5b+5c+5d+9}) \cup E(K_{5b,5c+5d+8}) \cup E(K_{5c,5d+7,}) \cup E(K_{5d,6}) \cup E(K_{1,2,3,4})$

Case (viii):Let k= 5a+1, l= 5b+4, m= 5c+4 and n= 5d+4
$E(K_{5a+1,5b+4,5c+4,5d+4}) = E(K_{5a,5b+5c,5d+12}) \cup E(K_{5b,5c+5d+9}) \cup E(K_{5c,5d+9}) \cup E(K_{5d,9}) \cup E(K_{1,4,4,4})$

Case (ix): Let k= 5a+2, l= 5b+3, m= 5c+3 and n= 5d+3
$E(K_{5a+2,5b+2,5c+2,5d+3}) = E(K_{5a,5b+5c+5d+9}) \cup E(K_{5a+1,5b+4,5c+4,5d+4}) \cup E(K_{5b,5c+5d+8}) \cup E(K_{5c,5d+8}) \cup E(K_{5d,8}) \cup E(K_{2,3,3,3})$

Case (x): Let k= 5a+2, l= 5b+2, m= 5c+2 and n= 5d+3
$E(K_{5a+2,5b+3,5c+3,5d+3}) = E(K_{5a,5b+5c+5d+9}) \cup E(K_{5b,5c+5d+8}) \cup E(K_{5c,5d+8}) \cup E(K_{5d,8}) \cup E(K_{2,2,2,3})$

In all cases it is possible to partition E ($K_{k,l,m,n}$) into the disjoint union of edge sets of $S_3 \cup S_4$ decomposable graphs. Then Lemma 3.1 and theorems 3.2 & 3.3 completes the proof.

4. CONLUSION

A complete characterization of complete four-partite graphs admitting an $S_3 \cup S_4$ -decomposition is obtained. These results extend the theory of star decompositions and contribute to the study of graph decompositions into disconnected trees.

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